§ 5.3 PRESENT VALUE OF AN ANNUITY ; AMORTIZATION

ORDINARY ANNUITY : e.g. \$500 DEPOSITED EVERY YEAR AT 6% INTEREST COMPOUNDED ANNUALLY FOR 10 YEARS.

THIS TURNS INTO $S_n = \dfrac{a(r^n - 1)}{r - 1}$ $\quad a = 500$, $n = 10$, $r = 1.06$

$$S_{10} = \frac{500(1.06^{10} - 1)}{.06} \approx \underline{\underline{6590.40}}$$

THE <u>PRESENT VALUE OF AN ANNUITY</u> IS THE ONE "LUMP SUM" DEPOSIT THAT, IF MADE TODAY, WOULD TURN INTO THE EQUIVALENT AMOUNT AT THE END OF THE ANNUITY.

$$P r^n = S_n \longrightarrow P = S_n r^{-n}$$

$$P r^n = \frac{a(r^n - 1)}{r - 1}$$

$$P = \frac{a(r^n - 1)r^{-n}}{r - 1} \rightarrow \boxed{P = \frac{a(1 - r^{-n})}{r - 1}}$$

e.g. $P = 6590.40(1.06)^{-10} = 3679.82$ ROUNDING ERROR

$$P = \frac{500(1 - 1.06^{-10})}{.06} = 3680.04$$ MORE ACCURATE ☆

ANOTHER INTERPRETATION: IF YOU DEPOSIT \$3680.04 INTO AN ACCOUNT EARNING 6% INTEREST COMPOUNDED ANNUALLY, YOU CAN WITHDRAW \$500 AT THE END OF EACH YEAR FOR 10 YEARS.

ex. A car costs \$25,000. After down payment of \$5000, the balance will be paid off in 36 equal monthly payments with interest 8% per year. Find the amount of each payment.

$$P = \frac{a(1-r^{-n})}{r-1}$$

$P = 20{,}000$

$r = 1 + \frac{.08}{12} \approx 1.0067$

$a =$ payment

$$20{,}000 = \frac{a\left(1-\left(1+\frac{.08}{12}\right)^{-36}\right)}{\frac{.08}{12}}$$

$$a = \frac{\frac{.08}{12}\cdot 20{,}000}{1-\left(1+\frac{.08}{12}\right)^{-36}} = \underline{\underline{626.73}}$$

$(626.73 \times 36 = 22{,}562.28)$

Note: $20{,}000\left(1+\frac{.08}{12}\right)^{36} \approx \dfrac{626.73\left(\left(1+\frac{.08}{12}\right)^{36}-1\right)}{\frac{.08}{12}}$ $\quad (Pr^n \approx S_n)$

$25{,}404.74 \approx 25{,}404.85$

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ex. Suppose you want to withdraw \$100,000 a year, every year for 20 years. How much do you need to deposit into an account earning 5% interest compounded annually to do this?

$$P = \frac{a(1-r^{-n})}{r-1}$$

$a = 100{,}000$

$r = 1.05$

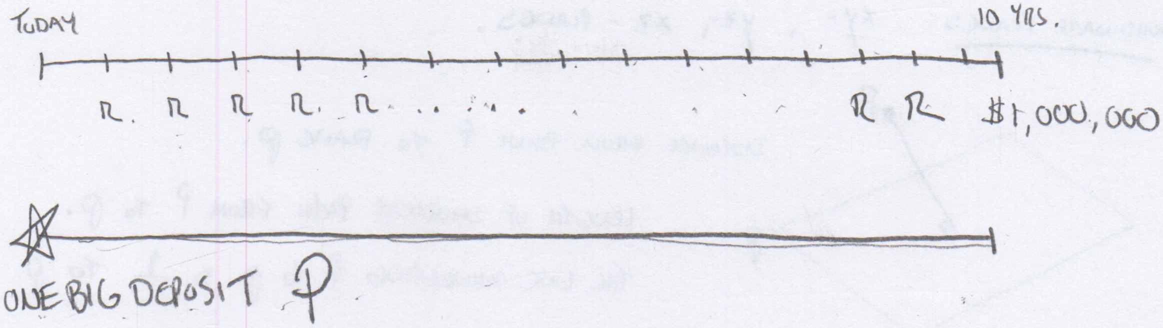
$n = 20$

$$P = \frac{100{,}000\left(1-(1.05)^{-20}\right)}{.05} = \boxed{1{,}246{,}221.03}$$

§5.3 PRESENT VALUE OF AN ANNUITY ; AMORTIZATION

THE PRESENT VALUE OF AN ANNUITY IS THE ONE "LUMP SUM" DEPOSIT THAT IF MADE TODAY WOULD TURN INTO THE FUTURE VALUE OF THE ANNUITY AT THE END OF THE TERM.

① ONE BIG DEPOSIT
↓ AT THE END
$1,000,000



FUTURE VALUE OF AN ANNUITY : $S_n = \dfrac{R\left[(1+i)^n - 1\right]}{i}$

COMPOUND AMOUNT $A = P(1+i)^n$ $\qquad \left(A = P\left(1+\frac{r}{n}\right)^{nt}\right)$

$i = \dfrac{\text{INTEREST RATE}}{\text{\# COMPOUNDS PER YEAR}}$

n = # OF PAYMENT/INTEREST PERIODS.

$20/WEEK FOR 624 WEEKS → $16,010.18

NOW, TO HAVE $16,010.18 IN 624 WEEKS WE COULD DEPOSIT P NOW AND LET IT SIT FOR 624 WEEKS.

$$P\left(1+\frac{.04}{52}\right)^{624} = 16,010.18 \rightarrow P = \frac{16,010.18}{\left(1+\frac{.04}{52}\right)^{624}} = \$9,908.62$$

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ex. YOU SET ASIDE \$20 A WEEK INTO AN ACCOUNT EARNING 4% INTEREST, COMPOUNDED WEEKLY. AT THE END OF 12 YEARS, HOW MUCH DO YOU HAVE?

(\$12480 TOTAL DEPOSITS) ← $Rn = (20)(624)$

$$S_n = \frac{R\left[(1+i)^n - 1\right]}{i}$$

$i = \frac{.04}{52} =$

$n = 12 \cdot 52 = 624$

$$= \frac{20\left(\left(1+\left(\frac{.04}{52}\right)\right)^{624} - 1\right)}{\left(\frac{.04}{52}\right)} = \boxed{\$16{,}010.18}$$

ex. A car costs \$25,000. After down payment of \$5,000, the balance will be paid off in 36 equal monthly payments with 8% interest per year. Find the amount of each payment.

$$S_n = \frac{R\left[\left(1+\frac{.08}{12}\right)^{36} - 1\right]}{\frac{.08}{12}} = P\left(1+\frac{.08}{12}\right)^{36}$$

$$(1+i)^{-n}\,\frac{R\left[(1+i)^n - 1\right]}{i} = P\,\cancel{(1+i)^n}\,\cancel{(1+i)^{-n}}$$

$$P = \frac{R\left[(1+i)^n - 1\right](1+i)^{-n}}{i}$$

$$\boxed{P = \frac{R\left[1 - (1+i)^{-n}\right]}{i}}$$

PRESENT VALUE OF ANNUITY

$$20000 = \frac{R\left[1 - \left(1+\frac{.08}{12}\right)^{-36}\right]}{\frac{.08}{12}}$$

$$\frac{(20000)\left(\frac{.08}{12}\right)}{\left[1 - \left(1+\frac{.08}{12}\right)^{-36}\right]} = R = \$626.73$$

ex. ALICE AGREES TO DEPOSIT \$100 INTO AN ACCOUNT EARNING 4% INTEREST AT THE END OF EVERY MONTH FOR 5 YEARS. SHE WILL THEN DONATE THIS MONEY TO CHARITY. $\Big\}$ ANNUITY

BOB IS GOING TO DEPOSIT ONE LUMP-SUM NOW INTO AN ACCOUNT EARNING 4% INTEREST COMPOUNDED MONTHLY FOR 5 YEARS. THEN DONATE THIS MONEY TO CHARITY. $\Big\}$ PRESENT VALUE

HOW MUCH SHOULD BOB DEPOSIT TO MATCH ALICE'S DONATION?

$$\boxed{P = \frac{R\left[1-(1+i)^{-n}\right]}{i}}$$

n = # PAYMENT PERIODS

$i = \dfrac{\text{ANNUAL INTEREST RATE}}{\text{\# PAYMENT PERIODS PER YEAR}}$

$$P = \frac{100\left[1-\left(1+\frac{.04}{12}\right)^{-60}\right]}{\left(\frac{.04}{12}\right)}$$

$i = \frac{r}{n}$

$= \boxed{\$5429.91}$

BOB DEPOSITS ON DAY 1.

$$\boxed{S_n = \frac{R\left[(1+i)^n-1\right]}{i}}$$

ALICE DEPOSITS TOTAL OF \$6,000.

HOW MUCH DO THEY EACH DONATE TO CHARITY?

FUTURE VALUE OF ALICE'S ANNUITY:

$$S_n = \frac{R\left[(1+i)^n - 1\right]}{i} = \frac{100\left[\left(1+\frac{.04}{12}\right)^{60} - 1\right]}{\frac{.04}{12}}$$

$$= \boxed{\$6,629.90}$$

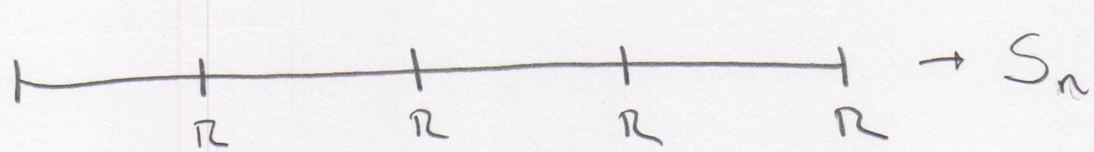
COMPOUND AMOUNT FOR BOB'S DEPOSIT:

$$A = P(1+i)^n = 5,429.91\left(1+\frac{.04}{12}\right)^{60}$$

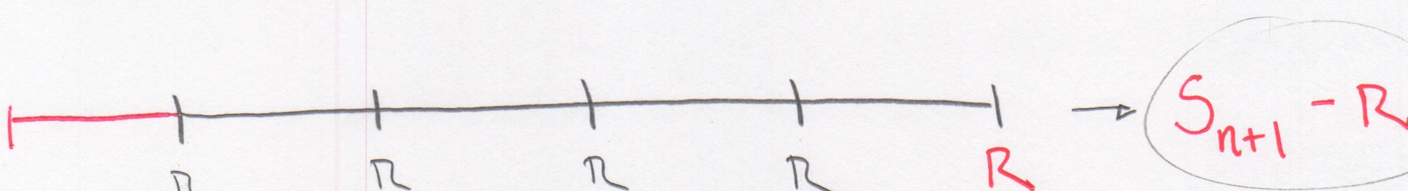
$$= \boxed{\$6,629.90}$$

✓

ANNUITY WITH n PAYMENTS AT END OF PAYMENT PERIOD



ANNUITY WITH n PAYMENTS AT BEGINNING OF PAYMENT PERIOD



"ANNUITY DUE"

FUTURE VALUE OF ANNUITY WHEN PAYMENTS MADE AT BEGINNING OF EACH PAYMENT PERIOD $= \dfrac{R\left[(1+i)^{n+1} - 1\right]}{i} - R$

AMORTIZATION TABLE

PAYMENT #	INTEREST	PAYMENT	AMOUNT OF PAYMENT PUT TOWARD PRINCIPAL	PRINCIPAL
0	-	-	-	100,000
1	500	1,110.21	610.21	99,389.79
2	496.95	1,110.21	613.26	98,776.53
3	493.88	1,110.21	616.33	98,160.20
			⋮	

$\times i$

$\times \frac{.06}{12}$

PAYMENT - INTEREST

- PAYMENT TOWARD PRINCIPAL

FUTURE VALUE OF ANNUITY: $S_n = \dfrac{R\left[(1+i)^n - 1\right]}{i}$

$\left(\begin{array}{l}\text{FUTURE VALUE OF A} \\ \text{LUMP-SUM-DEPOSIT}\end{array}\right)$

SET EQUAL

COMPOUND AMOUNT $A = P(1+i)^n$

$i = \dfrac{\text{ANNUAL INTEREST RATE}}{\text{\# COMPOUNDS PER YEAR}}$

n = # PAYMENT PERIODS

(# COMPOUND PERIODS)

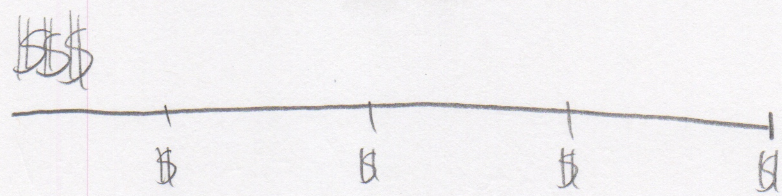
$$\cancel{(1+i)^{-n}} \cdot P\cancel{(1+i)^n} = \frac{R\left[(1+i)^n - 1\right]}{i}(1+i)^{-n}$$

$$\boxed{P = \frac{R\left[1 - (1+i)^{-n}\right]}{i}}$$

AMORTIZATION

A LOAN IS AMORTIZED WHEN BOTH THE PRINCIPAL & THE INTEREST ARE PAID BY A SEQUENCE OF EQUAL PERIODIC PAYMENTS.

e.g. MORTGAGE, CAR PAYMENTS,



AMORTIZATION TABLE

YOU ARE LOANED \$100,000 AT 6% INTEREST.

YOU WANT TO MAKE EQUAL-SIZE MONTHLY PAYMENTS,

AND HAVE LOAN PAID OFF IN 10 YEARS

(AMORTIZE THE LOAN INTO 120 PAYMENTS).

$$P = \frac{R\left[1-(1+i)^{-n}\right]}{i}$$

$$\left(\frac{.06}{12}\right)100{,}000 = \frac{R\left[1-\left(1+\frac{.06}{12}\right)^{(-120)}\right]}{\cancel{\left(\frac{.06}{12}\right)}}\cancel{\left(\frac{.06}{12}\right)}$$

$$R = \frac{(100{,}000)\left(\frac{.06}{12}\right)}{\left[1-\left(1+\left(\frac{.06}{12}\right)\right)^{(-120)}\right]} = \boxed{\$1{,}110.21}$$