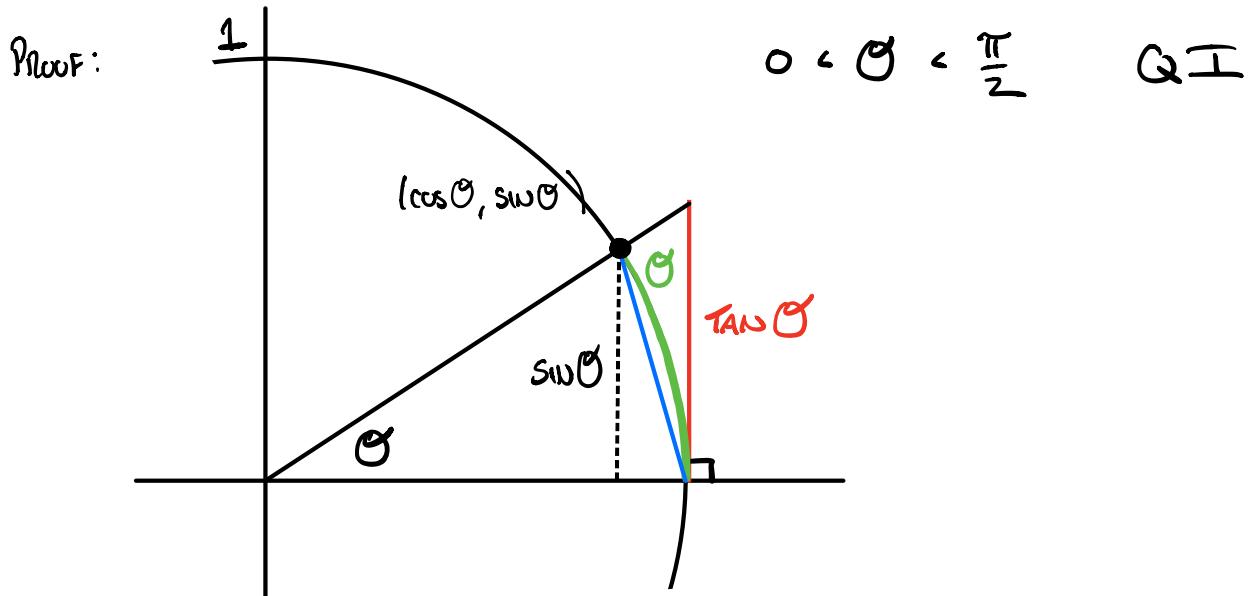


§ 2.4 Derivatives of Trig Functions

Two useful limits:

$$(1) \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$



$$\text{AREA } \triangle \text{ (blue)} < \text{AREA } \text{sector} \text{ (green)} < \text{AREA } \triangle \text{ (red)}$$

$$\frac{1}{2}(1)(\sin \theta) < \pi r^2 \left(\frac{\theta}{2\pi} \right) \quad (r=1) < \frac{1}{2}(1)(\tan \theta)$$

$$\Rightarrow \sin \theta < \theta < \tan \theta, \quad 0 < \theta < \frac{\pi}{2}$$

$$1 < \frac{\theta}{\sin \theta} < \frac{\sin \theta}{\cos \theta} \cdot \frac{1}{\sin \theta}$$

Squeeze Thm $\Rightarrow \lim_{\theta \rightarrow 0^+} \underbrace{\frac{1}{\sin \theta}}_1 \leq \lim_{\theta \rightarrow 0^+} \frac{\theta}{\sin \theta} \leq \lim_{\theta \rightarrow 0^+} \underbrace{\frac{1}{\cos \theta}}_1$

AND SO ALSO $\lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta} = 1$

$$\text{SINCE } \frac{\sin(-\theta)}{-\theta} = -\frac{\sin \theta}{-\theta} = \frac{\sin \theta}{\theta}$$

$$\text{WE HAVE } \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

ENGINEERING
FOR θ IS SMALL
 $\sin \theta \approx \theta$

$$(i) \lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} = 0$$

PYTHAGOREAN IDENTITY.

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\text{Proof: } \lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} \cdot \frac{\cos \theta + 1}{\cos \theta + 1}$$

$$= \lim_{\theta \rightarrow 0} \frac{\cos^2 \theta - 1}{\theta(\cos \theta + 1)} = \lim_{\theta \rightarrow 0} \frac{-\sin^2 \theta}{\theta(\cos \theta + 1)}$$

$$= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \lim_{\theta \rightarrow 0} \frac{-\sin \theta}{\cos \theta + 1}$$

$$\therefore 1 \cdot 0 = 0$$

□

$$\frac{d}{dx} [\sin x] = \cos x$$

$$\text{let } f(x) = \sin(x)$$

$$\sin x := \sin(x)$$

$$\text{Proof: } \frac{d}{dx} [\sin x] = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h}$$

ADDITION
FORMULA FOR
 \sin

$$= \lim_{h \rightarrow 0} \frac{\sin(x) \cos(h) + \sin(h) \cos(x)}{h} - \sin x$$

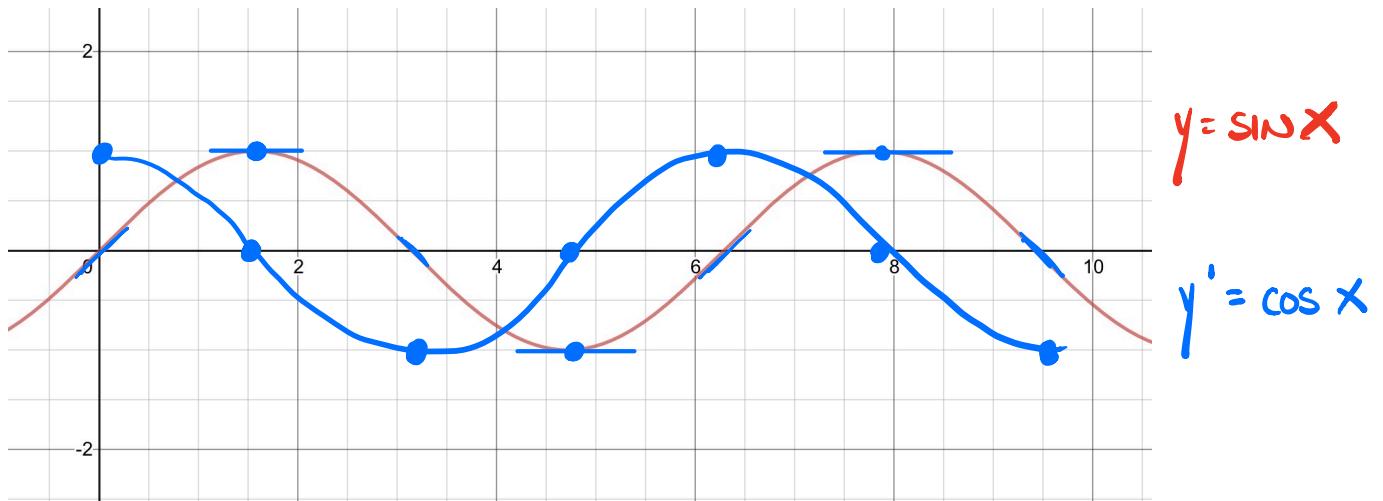
$$= \lim_{h \rightarrow 0} \frac{\sin(h)}{h} \cos x + \sin(x) \frac{\cos(h) - 1}{h}$$

$$= \left(\lim_{h \rightarrow 0} \frac{\sin(h)}{h} \right) \left(\lim_{h \rightarrow 0} \cos x \right) + \left(\lim_{h \rightarrow 0} \sin(x) \right) \left(\lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h} \right)$$

$$= (1) \cos(x) + \sin(x)(0)$$

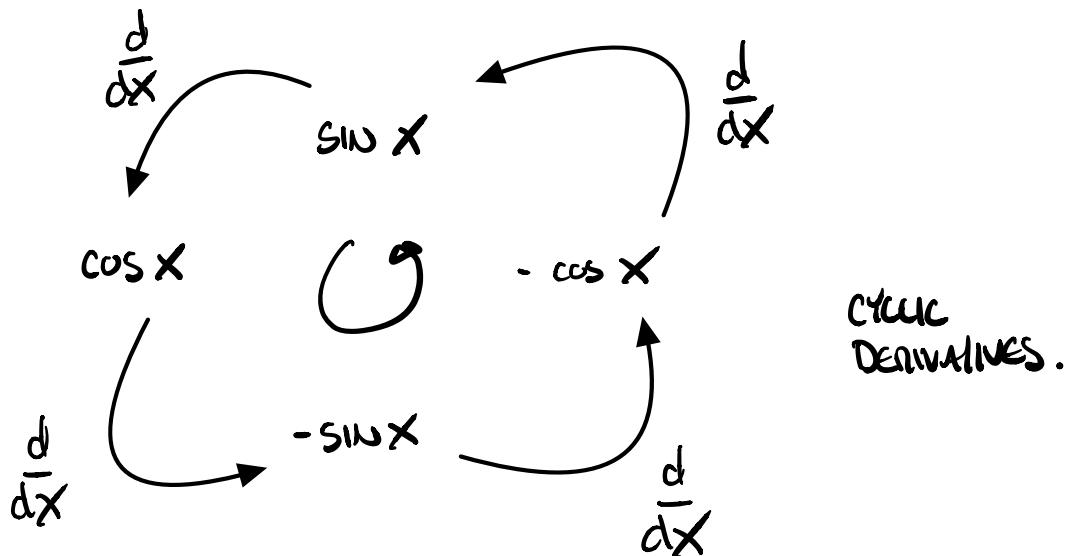
$$= \cos(x)$$

□



$$\frac{d}{dx} [\cos x] = -\sin x$$

Proof: Similar to proof that $\frac{d}{dx} \sin x = \cos x$.



$$\underline{\text{ex.}} \quad f(x) = x^2 \sin x$$

$$f'(x) = \frac{d}{dx} [x^2 \sin x] \quad \text{Product} \quad \frac{d}{dx}(fg) = f'g + fg'$$

$$= \frac{d}{dx}[x^2] \sin x + x^2 \frac{d}{dx}[\sin x]$$

$$= 2x \sin x + x^2 \cos x$$

$$\underline{\text{ex.}} \quad f(x) = \tan x = \frac{\sin x}{\cos x}$$

$$f'(x) = \frac{d}{dx} \left[\frac{\sin x}{\cos x} \right] = \frac{\cos x \frac{d}{dx} \sin x - \sin x \frac{d}{dx} \cos x}{(\cos x)^2}$$

$$= \frac{\cos x \cos x - \sin x (-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \left(\frac{1}{\cos x} \right)^2$$

$$= \sec^2 x \quad \frac{d}{dx} \tan x = \sec^2 x$$

$$\underline{\text{ex.}} \quad \frac{d}{dx} \left[\frac{1}{f(x)} \right] = \frac{f(x) \frac{d}{dx}[1] - 1 \frac{d}{dx}[f(x)]}{f(x)^2}$$

$$= \frac{-f'(x)}{f(x)^2}$$

$$\frac{d}{dx} \sec x = \sec x \tan x$$

$$\begin{aligned}\frac{d}{dx} \left[\frac{1}{\cos x} \right] &= \frac{-(-\sin x)}{(\cos x)^2} = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} \\ &= \tan x \sec x\end{aligned}$$

□

$$\frac{d}{dx} [\csc x] = -\csc x \cot x$$

$$\frac{d}{dx} [\cot x] = -\csc^2 x$$

$$\text{ex. } F(x) = \frac{\sec x}{1 + \tan x}$$

$$\begin{array}{lll} F(x) = \frac{f(x)}{g(x)} & f(x) = \sec x & g(x) = 1 + \tan x \\ & f'(x) = \sec x \tan x & g'(x) = 0 + \sec^2 x \end{array}$$

$$\text{Quotient Rule: } F'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

$$\frac{\sec x \tan x}{1 + \tan x} + \frac{\sec x \tan^2 x}{(1 + \tan x)^2}$$

$$F'(x) = \frac{(1 + \tan x)(\sec x \tan x) - \sec x \sec^2 x}{(1 + \tan x)^2}$$

$$= \sec x (\tan x + \tan^2 x - \sec^2 x)$$

$$(1 + \tan x)^2$$

$$\sin^2 x + \cos^2 x = 1$$

PYTH. ID. TRUE FOR ALL x

$$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$$

$$1 + \cot^2 x = \csc^2 x$$

$$\frac{\sec x (\tan x - 1)}{(1 + \tan x)^2}$$

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\tan^2 x + 1 = \sec^2 x$$

$$\tan^2 x - \sec^2 x = -1$$

$$F'(x) = \frac{(1 + \tan x)(\sec x \tan x) - \sec x \sec^2 x}{(1 + \tan x)^2}$$

$$= \frac{(1 + \cancel{\tan x})(\sec x \cancel{\tan x})}{(1 + \tan x)^2} - \frac{\sec x \sec^2 x}{(1 + \tan x)^2}$$

$$= \frac{\sec x \tan x}{1 + \tan x} - \frac{\sec^3 x}{(1 + \tan x)^2}$$

§ 2.5 THE CHAIN RULE

Let $y = f(u)$ & $u = g(x)$

(so $y = f(g(x))$, function composition)

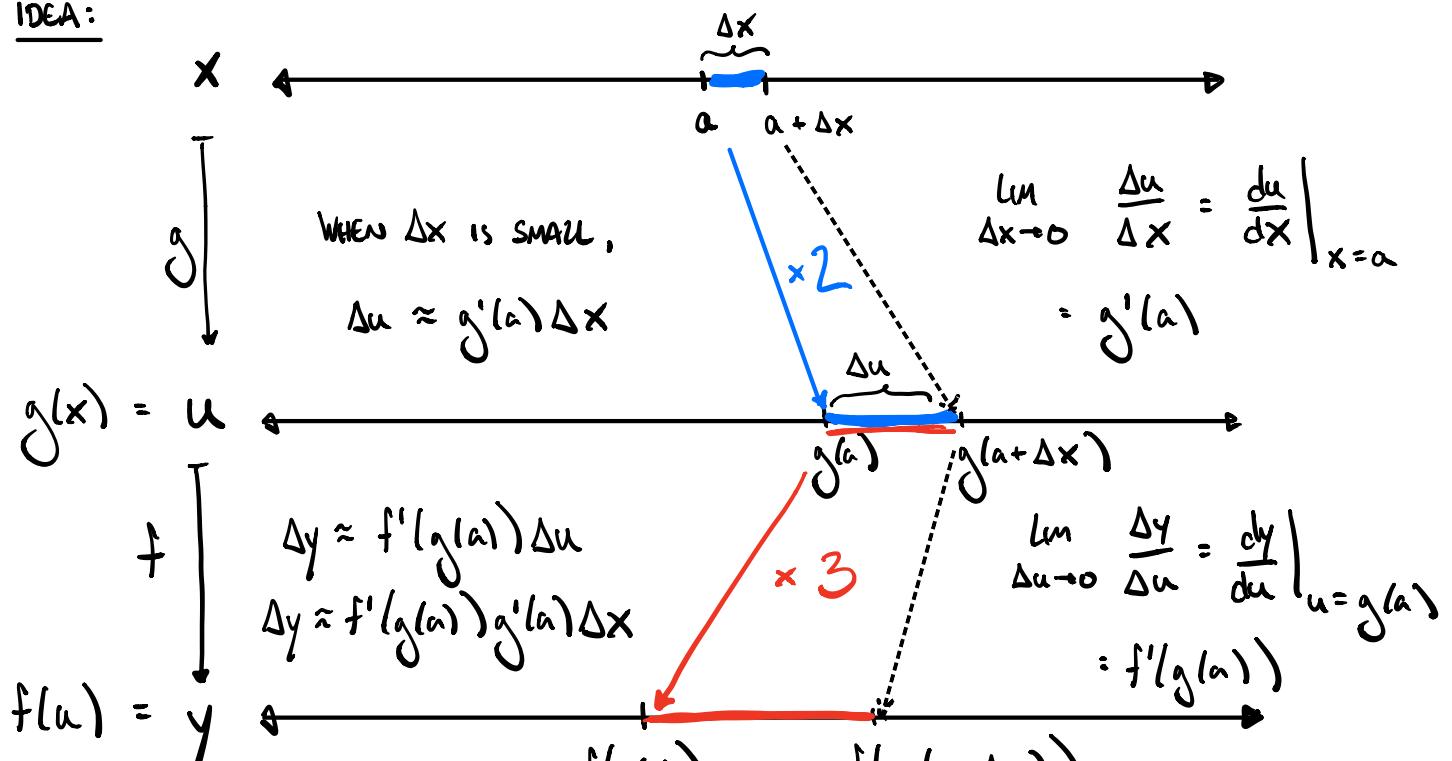
CHAIN RULE: IF $y = f(u)$ & $u = g(x)$

THEN $\frac{dy}{dx} = f'(u) g'(x)$

$\frac{dy}{dx} = f'(g(x)) g'(x)$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

IDEA:



$$f(g(a)) \quad f(g(a + \Delta x)) \\ = f(g(a) + \Delta u)$$

$\frac{\Delta y}{\Delta x} \approx f'(g(a))g'(a)$ WHEN Δx is small

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = f'(g(a))g'(a).$$

ex. EVALUATE $\frac{d}{dx} \sin(x^2) = \frac{d}{dx} f(g(x))$

where $f(x) = \sin x \quad g(x) = x^2$
 $f'(x) = \cos x \quad g'(x) = 2x$

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

$$= \cos(g(x)) \cdot 2x \\ = \cos(x^2) \cdot 2x = \boxed{2x \cos x^2}$$

ex. $\frac{d}{dx} \left[\sqrt{\sin x + \cos x} \right]$

$$= \frac{d}{dx} [f(g(x))] \quad \text{where } f(x) = x^{\frac{1}{2}} \\ g(x) = \sin x + \cos x$$

$$f'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$g'(x) = \cos x - \sin x$$

$$\frac{d}{dx} [f(g(x))] = f'(g(x)) g'(x)$$

$$= \frac{1}{2\sqrt{g(x)}} g'(x) = \boxed{\frac{\cos x - \sin x}{2\sqrt{\sin x + \cos x}}}$$