

CHAIN RULE:

ex. $F(x) = \underbrace{\sqrt{x^2+6}}_{f(x)} \underbrace{\sin(3x-1)}_{g(x)} = f(x) g(x)$

$$F'(x) = f(x) \frac{d}{dx} [g(x)] + \frac{d}{dx} [f(x)] g(x)$$

$$= f(x) g'(x) + f'(x) g(x)$$

DEPENDENCE

$$y = f(x) = (x^2+6)^{\frac{1}{2}} \quad \text{let } u = x^2+6$$

$$y = u^{\frac{1}{2}}$$

$$\frac{dy}{du} = \frac{1}{2} u^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{2} u^{-\frac{1}{2}} (2x+0)$$

$$= \frac{1}{2} (x^2+6)^{-\frac{1}{2}} (2x)$$



$$f'(x) = x(x^2+6)^{-\frac{1}{2}}$$

$$g(x) = \underbrace{\sin(3x-1)}_u$$

$$g'(x) = \cos(3x-1) \cdot (3-0) \\ = 3 \cos(3x-1)$$

$$F'(x) = f(x) g'(x) + f'(x) g(x)$$

$$= (x^2+6)^{\frac{1}{2}} \cdot 3 \cos(3x-1) + x(x^2+6)^{-\frac{1}{2}} \sin(3x-1)$$

$$= (x^2+6)^{-\frac{1}{2}} \left[3(x^2+6)^{\frac{1}{2}} \cos(3x-1) + x \sin(3x-1) \right]$$

$$= \frac{3(x^2+6)\cos(3x-1) + x\sin(3x-1)}{\sqrt{x^2+6}}$$

ex.

$$y = \cos \sqrt{\sin(\tan \pi x)}$$

$$\begin{aligned} &= \cos \left(\sqrt{\sin(\tan(\pi x))} \right) \\ y' &= -\sin \left(\sqrt{\sin(\tan(\pi x))} \right) \cdot \frac{1}{2} (\sin(\tan(\pi x)))^{-\frac{1}{2}} \\ &\quad \cdot \cos(\tan(\pi x)) \cdot \sec^2(\pi x) \cdot \pi \end{aligned}$$

§ 2.6 IMPLICIT DIFFERENTIATION

WARMUP WITH PRODUCT RULE / CHAIN RULE:

ex. Suppose f is a DIFFERENTIABLE function.

$$\text{Let } y = f(x) \quad ; \quad \text{Let } w = x^2 f(x) = x^2 y$$

(Note that w is a function of x)

$$\text{THEN } \frac{dw}{dx} = w' = \frac{d}{dx}[x^2]f(x) + x^2 \frac{d}{dx}[f(x)]$$

Product rule

$$w' = 2x f(x) + x^2 f'(x) , \quad \text{or}$$

$$w' = 2xy + x^2 y'$$

ex. Suppose f is a DIFFERENTIABLE function.

$$\text{Let } y = f(x) \quad ; \quad \text{Let } w = x^2 f(x)^3 = x^2 y^3$$

THEN $\frac{dw}{dx} = w' = \frac{d}{dx}[x^2]f(x)^3 + x^2 \frac{d}{dx}[f(x)^3]$

PRODUCT RULE

$$w' = 2x f(x)^3 + x^2 \cdot 3f(x)^2 f'(x) \text{ , or CHAIN RULE}$$

$$w' = 2xy^3 + 3x^2y^2y'$$

IMPLICIT DIFFERENTIATION

WHEN WE WRITE $y = f(x)$

e.g. $y = -\sqrt{1-x^2}$ (EXPLICIT EQUATION)

↑ ALL x's, NO y's

SOLVED FOR y

WE SAY y IS AN EXPLICIT FUNCTION OF x .

WHEN WE WRITE AN ARBITRARY EQUATION IN x & y ,

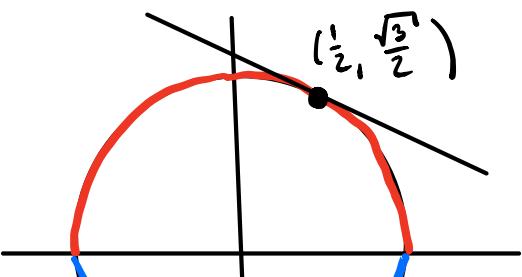
e.g. $x^2 + y^2 = 1$ (IMPLICIT EQ.)

WE SAY y IS AN IMPLICIT FUNCTION OF x .

GIVEN A VALUE FOR x , WE CAN SOLVE FOR VALUE(S)

OF y THAT SATISFIES THE EQUATION.

e.g. $y = \begin{cases} \sqrt{1-x^2} & , y \geq 0 \\ -\sqrt{1-x^2} & , y \leq 0 \end{cases}$



NOTE THAT THE IMPLICIT EQ HAS MULTIPLE EXPLICIT FORMS.

To FIND THE TANGENT LINE TO
IMPLICIT CURVE $x^2 + y^2 = 1$
AT THE POINT $(\frac{1}{2}, \frac{\sqrt{3}}{2})$...

SLOPE OF THE TANGENT LINE IS $\frac{dy}{dx}$ AT THAT POINT.

$$\left. \frac{dy}{dx} \right|_{(\frac{1}{2}, \frac{\sqrt{3}}{2})}$$

To FIND $\frac{dy}{dx}$: (1) TAKE DERIVATIVES WITH RESPECT TO x ($\frac{d}{dx}$)
OF BOTH SIDES OF THE EQUATION
(2) DO SO UNDER THE ASSUMPTION THAT
 y IS A DIFFERENTIABLE FUNCTION / FUNCTIONS OF x .
(CHAIN RULE!)

SAY $y = f(x)$

$$(1) \quad x^2 + y^2 = 1$$

$$\frac{d}{dx}[x^2 + y^2] = \frac{d}{dx}[1]$$

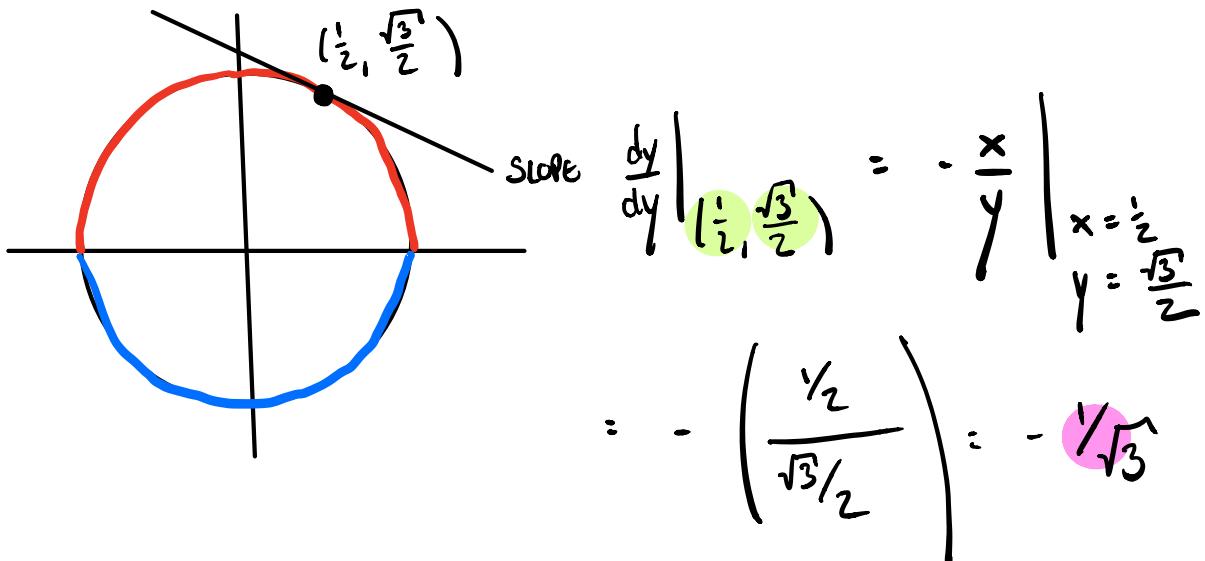
$$(2) \quad \frac{d}{dx}[x^2 + f(x)^2] = \frac{d}{dx}[1]$$

$$2x + 2f(x)f'(x) = 0$$

$$2x + 2y y' = 0$$

$$2y y' = -2x$$

$$y' = -\frac{2x}{2y} = -\frac{x}{y}$$



TANGENT LINE EQ: $y - \frac{\sqrt{3}}{2} = -\frac{1}{\sqrt{3}}(x - \frac{1}{2})$

5-20 Find dy/dx by implicit differentiation.

5. $x^2 - 4xy + y^2 = 4$

6. $2x^2 + xy - y^2 = 2$

7. $x^4 + x^2y^2 + y^3 = 5$

8. $x^3 - xy^2 + y^3 = 1$

9. $\frac{x^2}{x+y} = y^2 + 1$

10. $y^5 + x^2y^3 = 1 + x^4y$

11. $y \cos x = x^2 + y^2$

12. $\cos(xy) = 1 + \sin y$

ex. $x^4 + x^2y^2 + y^3 = 5$. FWD $\frac{dy}{dx}$, i.e. y' .

$$\frac{d}{dx} [x^4 + x^2y^2 + y^3] = \frac{d}{dx}[5]$$

$$\frac{d}{dx}[x^4] + \frac{d}{dx}[x^2y^2] + \frac{d}{dx}[y^3] = 0$$

$$4x^3 + \frac{d}{dx}[x^2]y^2 + x^2 \frac{d}{dx}[y^2] + \frac{d}{dx}[y^3] = 0$$

$$\frac{d}{dx}[f(x)^2] \quad \frac{d}{dy}[y^3] \frac{dy}{dx}$$

$$2 f(x) \cdot f'(x)$$

CHAIN RULE

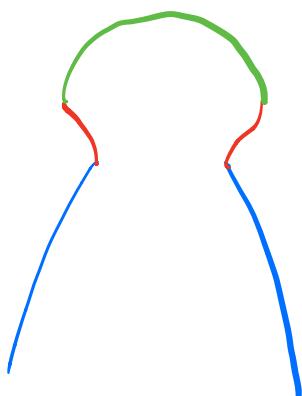
$$4x^3 + 2xy^2 + x^2 \cdot 2yy' + 3y^2y' = 0$$

Solve for y'

$$2x^2yy' + 3y^2y' = -4x^3 - 2xy^2$$

$$y'(2x^2y + 3y^2) = -4x^3 - 2xy^2$$

$$y' = \frac{-4x^3 - 2xy^2}{2x^2y + 3y^2}$$



ex. Let $y = x^{\frac{p}{q}}$, p, q are integers.

$$(y' = \frac{p}{q} x^{\frac{p}{q}-1})$$

$$(y)^q = (x^{\frac{p}{q}})^q$$

$$y^q = x^p \quad y \text{ is DIFF. FUNC. OF } x$$

$$\frac{d}{dx} [y^p] = \frac{d}{dx} [x^p]$$

$$\frac{d}{dy} [y^p] \cdot \frac{dy}{dx} = p x^{p-1}$$

$$p y^{p-1} \cdot \frac{dy}{dx} = p x^{p-1}$$

$$\frac{dy}{dx} = \frac{p x^{p-1}}{q y^{q-1}} = \frac{p}{q} \frac{x^{p-1}}{(x^{\frac{p}{q}})^{q-1}}$$

$$= \frac{p}{q} x^{(p-1) - (q - \frac{p}{q})}$$

$$= \frac{p}{q} x^{\frac{p}{q}-1} \quad \checkmark$$

25-32 Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

25. $y \sin 2x = x \cos 2y, \quad (\pi/2, \pi/4)$

IMPLICIT EQ.

$$\frac{d}{dx} [y \sin(2x)] = \frac{d}{dx} [x \cos(2y)]$$

REMEMBER
WE ASSUME $y = f(x)$

$$\frac{dy}{dx} \cdot \sin 2x + y \frac{d}{dx} [\sin(2x)] = \frac{d}{dx} [x] \cos(2y) + x \frac{d}{dx} [\cos(2y)]$$

$$\frac{dy}{dx} \sin 2x + y \cos(2x) \cdot 2 = \cos(2y) + x (-\sin(2y)) \cdot 2 \frac{dy}{dx}$$

$$\frac{dy}{dx} \left(\sin 2x + 2x \sin(2y) \right) = \cos(2y) - 2y \cos(2x)$$

$$\frac{dy}{dx} = \frac{\cos(2y) - 2y \cos(2x)}{\sin 2x + 2x \sin(2y)}$$

$$\begin{aligned}\frac{dy}{dx} \Big|_{\left(\frac{\pi}{2}, \frac{\pi}{4}\right)} &= \frac{\cos\left(2 \cdot \frac{\pi}{4}\right) \cdot 2 \cdot \frac{\pi}{4} \cos\left(2 \cdot \frac{\pi}{2}\right)}{\sin\left(2 \cdot \frac{\pi}{2}\right) + 2 \cdot \frac{\pi}{2} \sin\left(2 \cdot \frac{\pi}{4}\right)} \\ &= \frac{0 - \frac{\pi}{2}[-1]}{0 + \pi[1]} = \boxed{\frac{1}{2}}\end{aligned}$$

LINE THROUGH $(\frac{\pi}{2}, \frac{\pi}{4})$ WITH SLOPE $\frac{1}{2}$

$$y - \frac{\pi}{4} = \frac{1}{2}(x - \frac{\pi}{2})$$

34. (a) The curve with equation $y^2 = x^3 + 3x^2$ is called the **Tschirnhausen cubic**. Find an equation of the tangent line to this curve at the point $(1, -2)$.
- (b) At what points does this curve have horizontal tangents?
- (c) Illustrate parts (a) and (b) by graphing the curve and the tangent lines on a common screen.

(b.) Horizontal TANGENTS WHEN $\frac{dy}{dx} = 0$

$$\frac{d}{dx} [y^2] = \frac{d}{dx} [x^3 + 3x^2]$$

$$\frac{d}{dy} [y^2] \cdot \frac{dy}{dx} = \frac{d}{dx} [x^3 + 3x^2] \quad \text{CHAIN RULE}$$

$$2y \frac{dy}{dx} = 3x^2 + 6x$$

$$\frac{dy}{dx} = \frac{3x(x+2)}{2y} = 0$$

$$x=0 \Rightarrow y=0$$

$$(y^2 = x^3 + 3x^2)$$

Not Horizontal Tangent

$$x = -2$$

$$y^2 = (-2)^3 + 3(-2)^2$$

$$y^2 = -8 + 12 = 4$$

$$y = \pm 2$$

$$(-2, 2) \text{ & } (-2, -2)$$

13. $\sqrt{x+y} = x^4 + y^4$

14. $y \sin(x^2) = x \sin(y^2)$

15. $\tan(x/y) = x + y$

16. $xy = \sqrt{x^2 + y^2}$

17. $\sqrt{xy} = 1 + x^2y$

18. $x \sin y + y \sin x = 1$

19. $\sin(xy) = \cos(x + y)$

20. $\tan(x - y) = \frac{y}{1 + x^2}$