

## § 4.2 THE DEFINITE INTEGRAL (continued)

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x \quad \Delta x = \frac{b-a}{n} \quad x_i = a + i\Delta x$$

### Properties of the Definite Integral

When we defined the definite integral  $\int_a^b f(x) dx$ , we implicitly assumed that  $a < b$ . But the definition as a limit of Riemann sums makes sense even if  $a > b$ . Notice that if we reverse  $a$  and  $b$ , then  $\Delta x$  changes from  $(b - a)/n$  to  $(a - b)/n$ . Therefore

$$\int_b^a f(x) dx = - \int_a^b f(x) dx$$



If  $a = b$ , then  $\Delta x = 0$  and so

$$\int_a^a f(x) dx = 0$$

We now develop some basic properties of integrals that will help us to evaluate integrals in a simple manner. We assume that  $f$  and  $g$  are continuous functions.

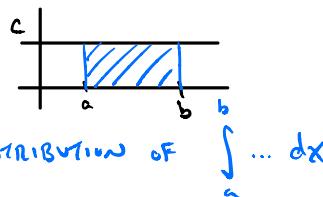
#### Properties of the Integral

$$1. \int_a^b c dx = c(b - a), \text{ where } c \text{ is any constant}$$

$$2. \int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$

$$3. \int_a^b cf(x) dx = c \int_a^b f(x) dx, \text{ where } c \text{ is any constant}$$

$$4. \int_a^b [f(x) - g(x)] dx = \int_a^b f(x) dx - \int_a^b g(x) dx$$



- [8]
  - [10]
  - [9]
  - [11]
- } CORRESPONDING PROPERTIES OF SUMS (ABOVE)

IN SUMMARY :  $\int_a^b [cf(x) + dg(x)] dx = c \int_a^b f(x) dx + d \int_a^b g(x) dx$

$a, d \in \mathbb{R}$

PROOF:

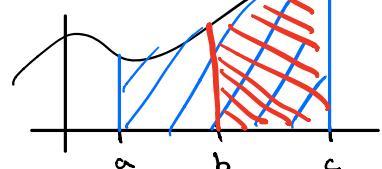
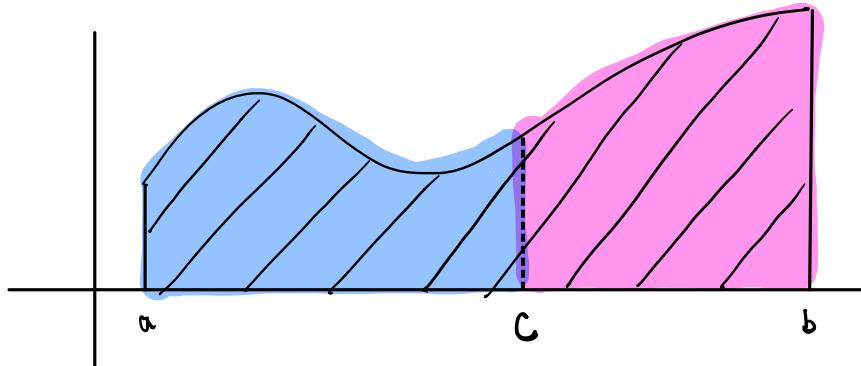
$$\int_a^b [cf(x) + dg(x)] dx = \lim_{n \rightarrow \infty} \left[ \sum_{i=1}^n \left[ (cf(x_i) + dg(x_i)) \Delta x \right] \right]$$

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} \left[ \left( \sum_{i=1}^n c f(x_i) + d g(x_i) \right) \Delta x \right] \\
 &= \lim_{n \rightarrow \infty} \left[ \sum_{i=1}^n c f(x_i) \Delta x + \sum_{i=1}^n d g(x_i) \Delta x \right] \\
 &= \lim_{n \rightarrow \infty} \sum_{i=1}^n c f(x_i) \Delta x + \lim_{n \rightarrow \infty} \sum_{i=1}^n d g(x_i) \Delta x \\
 &= c \underbrace{\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x}_{\int_a^b f(x) dx} + d \underbrace{\lim_{n \rightarrow \infty} \sum_{i=1}^n g(x_i) \Delta x}_{\int_a^b g(x) dx} \\
 &= c \int_a^b f(x) dx + d \int_a^b g(x) dx. \quad \square
 \end{aligned}$$

5.  $\int_a^c f(x) dx + \int_c^b f(x) dx = \int_a^b f(x) dx$

 +  = 

$y=f(x)$



$$\begin{aligned}
 &\int_a^c f(x) dx + \int_c^b f(x) dx \\
 &= \int_a^b f(x) dx - \int_b^c f(x) dx
 \end{aligned}$$

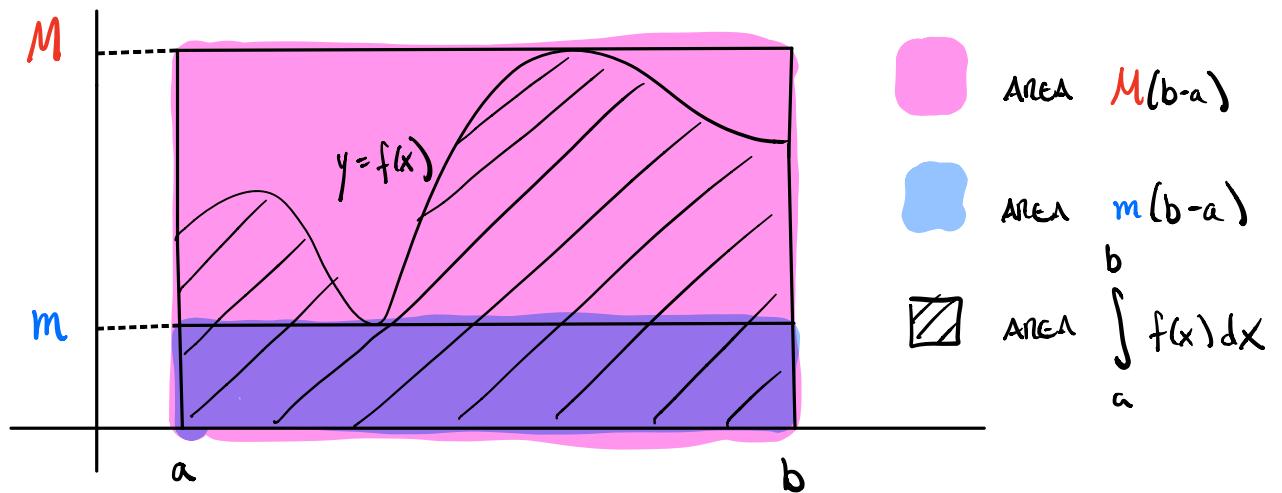
ex. If  $\int_3^{12} f(x) dx = -10$  &  $\int_8^{12} f(x) dx = 4$ , FIND  $\int_3^8 f(x) dx$ .

$$\begin{aligned}
 \int_3^8 f(x) dx + \int_8^{12} f(x) dx &= \int_3^{12} f(x) dx \\
 \int_3^8 f(x) dx &= \int_3^{12} f(x) dx - \int_8^{12} f(x) dx \\
 &= -10 - \frac{1}{2} \int_8^{12} 2f(x) dx \\
 &= -10 - \frac{1}{2}(4) = -12 \Rightarrow \int_3^8 3f(x) dx = 3 \int_3^{12} f(x) dx \\
 &= 3(-12) \\
 &= -36
 \end{aligned}$$

### Comparison Properties of the Integral

6. If  $f(x) \geq 0$  for  $a \leq x \leq b$ , then  $\int_a^b f(x) dx \geq 0$ .
7. If  $f(x) \geq g(x)$  for  $a \leq x \leq b$ , then  $\int_a^b f(x) dx \geq \int_a^b g(x) dx$ .
8. If  $m \leq f(x) \leq M$  for  $a \leq x \leq b$ , then

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$

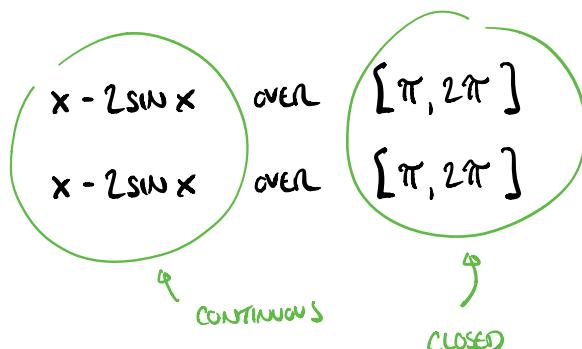


Ex. Use Midpoint  $\delta$  to estimate  $\int_{\pi}^{2\pi} x - 2 \sin x \, dx$

$$m(2\pi - \pi) \leq \int_{\pi}^{2\pi} x - 2 \sin x \, dx \leq M(2\pi - \pi)$$

WHERE  $m$  is ABS MIN VAL OF

$M$  is ABS MAX VAL OF



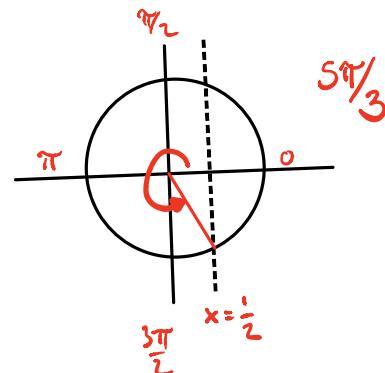
could occur at CRIT. PTS or ENDPOINTS:

Extreme Value Thm  $\Rightarrow m, M$  exist.

$$f(x) = x - 2 \sin x, \quad \pi \leq x \leq 2\pi$$

$$f'(x) = 1 - 2 \cos x = 0$$

$$\cos x = \frac{1}{2} \Rightarrow x = \frac{5\pi}{3}$$



x	$f(x) = x - 2 \sin x$
$\pi$	$\pi$ MIN $m$
$\frac{5\pi}{3}$	$\frac{5\pi}{3} - 2 \left( -\frac{\sqrt{3}}{2} \right) = \frac{5\pi}{3} + \sqrt{3}$ MAX $M$
$2\pi$	$2\pi$

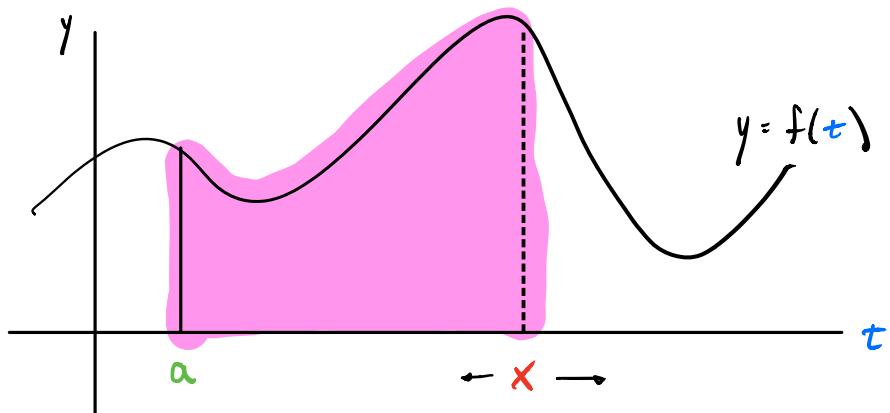
$$\pi(2\pi - \pi) \leq \int_{\pi}^{2\pi} x - 2 \sin x \, dx \leq \left( \frac{5\pi}{3} + \sqrt{3} \right) (2\pi - \pi)$$

## § 4.3 THE FUNDAMENTAL THEOREM OF CALCULUS

WE CAN USE THE DEFINITE INTEGRAL TO DEFINE FUNCTIONS.

CONSIDER THE "AREA SO FAR" FUNCTION

$$g(x) = \int_a^x f(t) dt$$

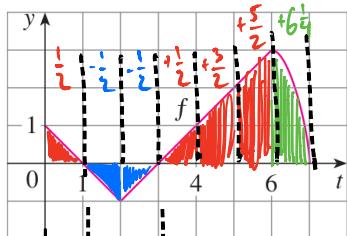


\* Note You can't HAVE THE SAME VARIABLE APPEAR IN INTEGAND & LIMITS OF INTEGRATION!

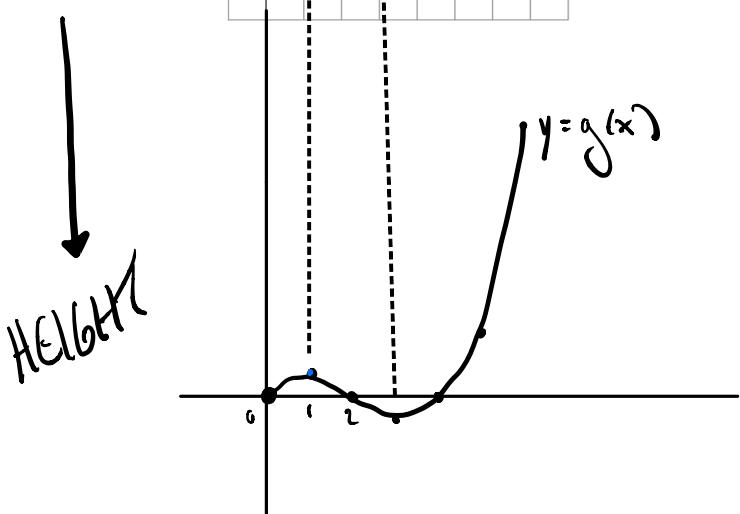
2. Let  $g(x) = \int_0^x f(t) dt$ , where  $f$  is the function whose graph is shown.

- (a) Evaluate  $g(x)$  for  $x = 0, 1, 2, 3, 4, 5$ , and  $6$ .
- (b) Estimate  $g(7)$ .
- (c) Where does  $g$  have a maximum value? Where does it have a minimum value?
- (d) Sketch a rough graph of  $g$ .

CUMULATIVE AREAS



$$2\frac{1}{4} \approx \frac{9}{4}$$



$$g(x) = \int_0^x f(t) dt \quad \longleftrightarrow \quad g'(x) = f(x)$$

$x$	$g(x) = \int_0^x f(t) dt$
0	$\int_0^0 f(x) dx = 0$
1	$\int_0^1 f(x) dx \approx -\frac{1}{2}$
2	$0$
3	$-\frac{1}{2}$
4	$0$
5	$\int_0^5 f(x) dx = \int_0^4 f(x) dx + \int_4^5 f(x) dx : \frac{3}{2}$
6	$4$
7	$4 + \frac{9}{4} = \frac{25}{4} \approx 6\frac{1}{4}$

$g(x)$  HAS MAX/MINs WHEN  $f(x) = 0$ .

**The Fundamental Theorem of Calculus, Part 1** If  $f$  is continuous on  $[a, b]$ , then the function  $g$  defined by

$$g(x) = \int_a^x f(t) dt \quad a \leq x \leq b$$

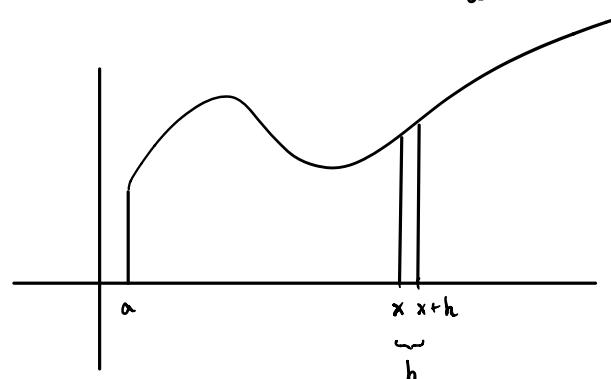
is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , and  $g'(x) = f(x)$ .

i.e.  $\frac{d}{dx} \int_a^x f(t) dt = f(x).$

**PROOF:** We must show that  $g'(x)$  exists for all  $x \in (a, b)$ , & equals  $f(x)$ .  
 (Continuity at  $x=a$  &  $x=b$  is not hard to see.)

For any  $x \in (a, b)$ , we have

$$\begin{aligned} g'(x) &= \lim_{h \rightarrow 0} \frac{1}{h} (g(x+h) - g(x)) = \lim_{h \rightarrow 0} \frac{1}{h} \left( \int_a^{x+h} f(t) dt - \int_a^x f(t) dt \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f(t) dt \end{aligned}$$



Extreme Value Thm  $\Rightarrow \exists u, v \in [x, x+h]$  such that ABS MAX VAL.

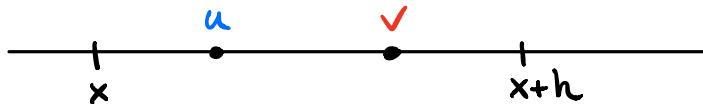
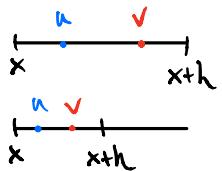
$$\text{ABS MIN VAL} \rightarrow f(u) \leq f(t) \leq f(v) \quad \forall t \in [x, x+h]$$

$$\Rightarrow f(u)h \leq \int_x^{x+h} f(t) dt \leq f(v)h \quad * \quad (\text{PROPERTY 8 OF DEF. INT.})$$

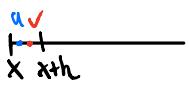
$$\Rightarrow f(u) \leq \frac{1}{h} \int_x^{x+h} f(t) dt \leq f(v)$$

\* ASSUMING  $f$  is positive  $\forall t \in [x, x+h]$  AND  $h > 0$ .  
 IF NOT THEN INEQUALITIES MAY NEED TO BE REVERSED. NO BIG DEAL.

$$\therefore f(u) \leq \frac{g(x+h) - g(x)}{h} \leq f(v)$$



$$x \leq u, v \leq x+h$$



AS  $h \rightarrow 0$  BOTH  $u \rightarrow x$   
 $v \rightarrow x$

$$\Rightarrow \lim_{h \rightarrow 0} f(u) \leq \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \leq \lim_{h \rightarrow 0} f(v)$$

$$\Rightarrow \underbrace{\lim_{u \rightarrow x} f(u)}_{f(x)} \leq \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \leq \underbrace{\lim_{v \rightarrow x} f(v)}_{f(x)}$$

$$\Rightarrow g'(x) = f(x) \quad \text{by Squeeze THM.}$$

□

IN GENERAL :  $\frac{d}{dx} \int_a^{g(x)} f(t) dt = f(g(x)) g'(x)$

PROOF : Let  $u = g(x)$ . Then  $\frac{d}{dx} \int_a^{g(x)} f(t) dt = \frac{d}{du} \int_a^u f(t) dt \frac{du}{dx}$

$$= f(u) \frac{du}{dx} = f(g(x)) g'(x) \quad \square$$

Note :  $\frac{d}{dx} \int_{m(x)}^{M(x)} f(t) dt = f(M(x)) M'(x) - f(m(x)) m'(x).$

PROOF :

$$\begin{aligned} \frac{d}{dx} \int_{m(x)}^{M(x)} f(t) dt &= \frac{d}{dx} \left[ \int_{m(x)}^a f(t) dt + \int_a^{M(x)} f(t) dt \right] \\ &= \frac{d}{dx} \left[ \int_a^{M(x)} f(t) dt - \int_a^{m(x)} f(t) dt \right] \\ &= f(M(x)) M'(x) - f(m(x)) m'(x). \end{aligned}$$

**The Fundamental Theorem of Calculus, Part 2** If  $f$  is continuous on  $[a, b]$ , then

$$\int_a^b f(x) dx = F(b) - F(a) *$$

where  $F$  is any antiderivative of  $f$ , that is, a function  $F$  such that  $F' = f$ .

$$* \text{ NOTATION: } F(x) \Big|_a^b = F(b) - F(a)$$

Proof:

$$\text{let } g(x) = \int_a^x f(t) dt \quad (g(a) = 0)$$

$$F(x) \Big|_a^b = F(b) - F(a)$$

$$[F(x)]_a^b = F(b) - F(a)$$

FTC 1  $\Rightarrow$   $g$  is an antiderivative of  $f$   
 $(g'(x) = f(x))$

obviously  $\int_a^b f(t) dt = \underbrace{g(b) - g(a)}_{(WHY?)} \quad (WHY?)$

$$\int_a^b f(t) dt - \underbrace{\int_a^a f(t) dt}_0 = \int_a^b f(t) dt$$

Now suppose  $F$  is an antiderivative of  $f$

then  $F(x) = g(x) + C$  for some  $C$  AND

$$F(b) - F(a) = g(b) + C - (g(a) + C) = g(b) - g(a) = g(b) = \int_a^b f(x) dx$$

□

$$\underline{\text{ex.}} \quad \text{FWO} \quad \int_1^2 \frac{(x^2 - 1)^2}{x^2} dx = \left. F(x) \right|_1^2 = F(2) - F(1)$$

$$= \int_1^2 \frac{x^4 - 2x^2 + 1}{x^2} dx = \int_1^2 x^2 - 2 + x^{-2} dx$$

$$= \left. \frac{1}{3}x^3 - 2x - x^{-1} \right|_1^2 \quad *$$

$$= \left( \frac{1}{3}(2)^3 - 2(2) - (2)^{-1} \right) - \left( \frac{1}{3}(1)^3 - 2(1) - (1)^{-1} \right)$$

$$= \frac{8}{3} - 4 - \frac{1}{2} - \frac{1}{3} + 2 + 1 = \frac{5}{6}$$

$$* = \frac{1}{3} [x^3]_1^2 - 2 [x]_1^2 - [x^{-1}]_1^2$$

$$= \frac{1}{3} (2^3 - 1^3) - 2(2-1) - (2^{-1} - 1^{-1})$$

$$= \frac{7}{3} - 2 - \left(-\frac{1}{2}\right) = \frac{5}{6}$$

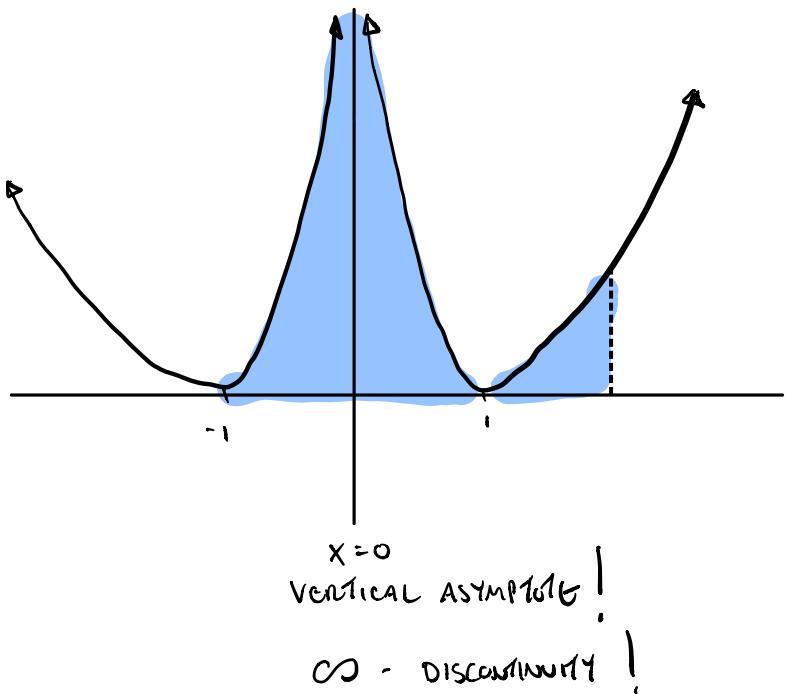
Ex. What is wrong with the following?

INTEGRAND  
CONTINUOUS  
ON THIS  
INTERVAL?

$$\left\{ \int_{-1}^2 \frac{(x^2 - 1)^2}{x^2} dx = \frac{1}{3}x^3 - 2x - x^{-1} \right|_{-1}^2$$

(No)

$$\begin{aligned} &= \frac{1}{3}(2^3 - (-1)^3) - 2(2 - (-1)) - (2^1 - (-1)^{-1}) \\ &= \frac{1}{3}(9) - 2(3) - \left(\frac{3}{2}\right) = -\frac{9}{2} \end{aligned}$$



$$\int_a^b f(x) dx = F(b) - F(a)$$

ONLY WHEN  $f$  IS CONTINUOUS  
ON  $[a, b]$  &  $F'(x) = f(x)$   
FOR ALL  $x \in (a, b)$ .

# More Practice?

\*

**7-18** Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of the function.

$$7. g(x) = \int_0^x \sqrt{t+t^3} dt$$

$$8. g(x) = \int_1^x \cos(t^2) dt$$

$$9. g(s) = \int_5^s (t-t^2)^8 dt$$

$$10. h(u) = \int_0^u \frac{\sqrt{t}}{t+1} dt$$

$$11. F(x) = \int_x^0 \sqrt{1+\sec t} dt$$

$$\left[ \text{Hint: } \int_x^0 \sqrt{1+\sec t} dt = - \int_0^x \sqrt{1+\sec t} dt \right]$$

$$12. R(y) = \int_y^2 t^3 \sin t dt$$

$$13. h(x) = \int_2^{1/x} \sin^4 t dt$$

$$14. h(x) = \int_1^{\sqrt{x}} \frac{z^2}{z^4+1} dz$$

$$15. y = \int_1^{3x+2} \frac{t}{1+t^3} dt$$

$$16. y = \int_0^{x^4} \cos^2 \theta d\theta$$

$$17. y = \int_{\sqrt{x}}^{\pi/4} \theta \tan \theta d\theta$$

$$18. y = \int_{\sin x}^1 \sqrt{1+t^2} dt$$

\*

**19-38** Evaluate the integral.

$$19. \int_1^3 (x^2 + 2x - 4) dx$$

$$20. \int_{-1}^1 x^{100} dx$$

$$21. \int_0^2 \left( \frac{4}{5}t^3 - \frac{3}{4}t^2 + \frac{2}{5}t \right) dt$$

$$22. \int_0^1 (1 - 8v^3 + 16v^7) dv$$

$$23. \int_1^9 \sqrt{x} dx$$

$$24. \int_1^8 x^{-2/3} dx$$

$$25. \int_{\pi/6}^{\pi} \sin \theta d\theta$$

$$26. \int_{-5}^5 \pi dx$$

$$27. \int_0^1 (u+2)(u-3) du$$

$$28. \int_0^4 (4-t)\sqrt{t} dt$$

$$29. \int_1^4 \frac{2+x^2}{\sqrt{x}} dx$$

$$30. \int_{-1}^2 (3u-2)(u+1) du$$

$$31. \int_{\pi/6}^{\pi/2} \csc t \cot t dt$$

$$32. \int_{\pi/4}^{\pi/3} \csc^2 \theta d\theta$$

$$17. y = \int_{\sqrt{x}}^{\pi/4} \theta \tan \theta d\theta$$

LET  $F(\theta)$  BE AN ANTIDERIVATIVE OF  $\theta \tan \theta$

$$F'(\theta) = \theta \tan \theta$$

$$\text{THEN } y = \int_{\sqrt{x}}^{\pi/4} \theta \tan \theta d\theta = F\left(\frac{\pi}{4}\right) - F(\sqrt{x})$$

$$\text{so } \frac{dy}{dx} = \frac{d}{dx} \left[ \underbrace{F\left(\frac{\pi}{4}\right)}_{\text{constant}} - F(\sqrt{x}) \right]$$

$$= 0 - F'(\sqrt{x}) \frac{d}{dx} [\sqrt{x}]$$

CHAIN RULE

$$= -F'(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$$

$$= -\cancel{\sqrt{x}} \tan \sqrt{x} \cdot \frac{1}{2\cancel{\sqrt{x}}}$$

$$= -\frac{1}{2} \tan \sqrt{x}$$