

Find the derivative of the function.

$$g(x) = \int_{\tan x}^{4x^2} \frac{1}{\sqrt{5+t^4}} dt$$

Let  $F(x)$  be antideriv. of  $\frac{1}{\sqrt{5+t^4}}$ .

$$g(x) = F(x) \Big|_{\tan x}^{4x^2} = F(4x^2) - F(\tan x)$$

$$g'(x) = \frac{d}{dx} [F(4x^2) - F(\tan x)]$$

$$= F'(4x^2) \cdot 8x - F'(\tan x) \cdot \sec^2 x$$

$$= \frac{1}{\sqrt{5+(4x^2)^4}} \cdot 8x - \frac{1}{\sqrt{5+(\tan x)^4}} \cdot \sec^2 x$$

Use the [definition](#) to find an expression for the area under the graph of  $f$  as a limit. Do not evaluate the limit.

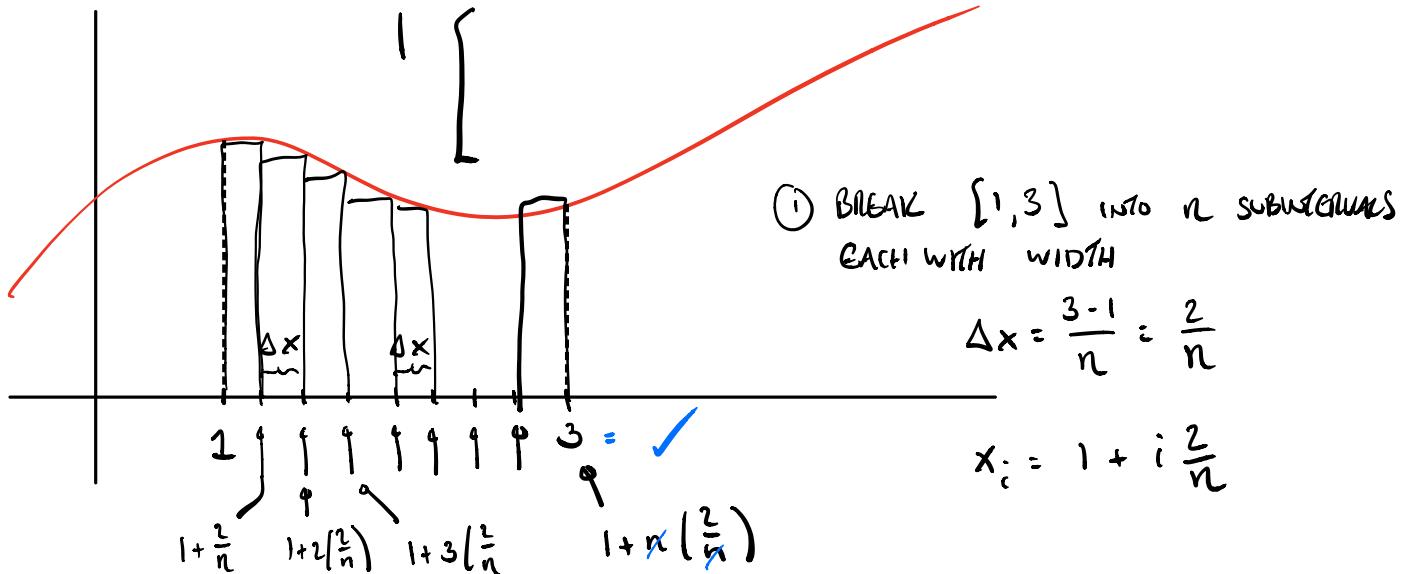
$$f(x) = \frac{8x}{x^2 + 6}, \quad 1 \leq x \leq 3$$

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \left( \boxed{\text{ } \cdot \text{ }} \right)$$

$$\left[ \frac{8(1+2i/n)}{(1+2i/n)^2+6} \cdot \frac{2}{n} \right]$$

$$\text{AREA} = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$$

$$\Delta x = \frac{b-a}{n} \quad x_i = a + i \Delta x$$



$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left( f\left(1 + i \frac{2}{n}\right) \right) \frac{2}{n}$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left( f\left(\frac{n+2i}{n}\right) \right) \frac{2}{n}$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left( \frac{8\left(\frac{n+2i}{n}\right)}{\left(\frac{n+2i}{n}\right)^2 + 6} \right) \frac{2}{n}$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left( \frac{8\left(\frac{n+2i}{n}\right)}{\left(\frac{n+2i}{n}\right)^2 + 6} \cdot \frac{n^2}{n^2} \right) \frac{2}{n}$$

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{8n(n+2i)}{(n+2i)^2 + 6n^2} \cdot \frac{2}{n} \\ &\boxed{\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{16(n+2i)}{(n+2i)^2 + 6n^2}} \end{aligned}$$

# §6.1 Inverse Functions

## I. Precalculus

- one-to-one functions
- Horizontal Line Test
- inverse functions
- graphs

## II Calculus

- Inverse Function Theorem

**1 Definition** A function  $f$  is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever } x_1 \neq x_2$$

"**Different Inputs**

**Produce Different Outputs**"

"**If You HAVE Equal Outputs  
Then You MUST HAVE Started With Equal Inputs**"

contrapositive  
statements

$$(P \Rightarrow Q) \quad (\neg Q \Rightarrow \neg P)$$

ex.  $f(x) = 3 + 4x$  is 1:1

But  $f(x) = x^2 + 2x + 3$  is not 1:1

**IF Equal Outputs:**  $f(a) = f(b)$

**EQUAL OUTPUTS**  
IF  $f(a) = f(b)$ , THEN

$$3 + 4a = 3 + 4b$$

$$4a = 4b$$

$$a = b$$

**EQUAL INPUTS**

Completing  
the sq.

$$\begin{aligned} a^2 + 2a + 3 &= b^2 + 2b + 3 \\ (a+1)^2 + 2 &= (b+1)^2 + 2 \\ (a+1)^2 &= (b+1)^2 \end{aligned}$$

$$\sqrt{(a+1)^2} = \sqrt{(b+1)^2}$$

$$|a+1| = |b+1|$$

$$\begin{aligned} x^2 &= 9 \\ x &= \pm 3 \end{aligned}$$

$$\sqrt{x^2} = \sqrt{3^2}$$

$$|x| = |3|$$

$$x = \pm 3$$

$$a+1 = \pm (b+1)$$

$$a = -1 \pm (b+1) = \begin{cases} b \\ -b-2 \end{cases}$$

a MIGHT NOT EQUAL b!

**Horizontal Line Test** A function is one-to-one if and only if no horizontal line intersects its graph more than once.

$$y = f(x)$$



**1 Definition** A function  $f$  is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever } x_1 \neq x_2$$

DIFF. y-coord.      DIFF. x-coord.

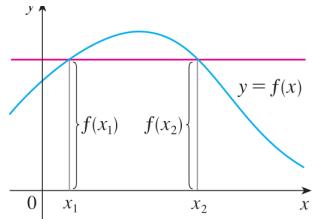
## 1:1 FUNCTIONS HAVE INVERSE FUNCTIONS!

**2 Definition** Let  $f$  be a one-to-one function with domain  $A$  and range  $B$ . Then its **inverse function**  $f^{-1}$  has domain  $B$  and range  $A$  and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

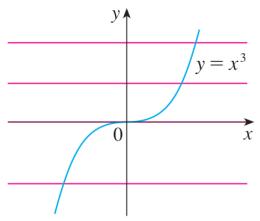
for any  $y$  in  $B$ .

$$f^{-1}(x) = y \iff f(y) = x$$



**FIGURE 2**

This function is not one-to-one because  $f(x_1) = f(x_2)$ .



**FIGURE 3**

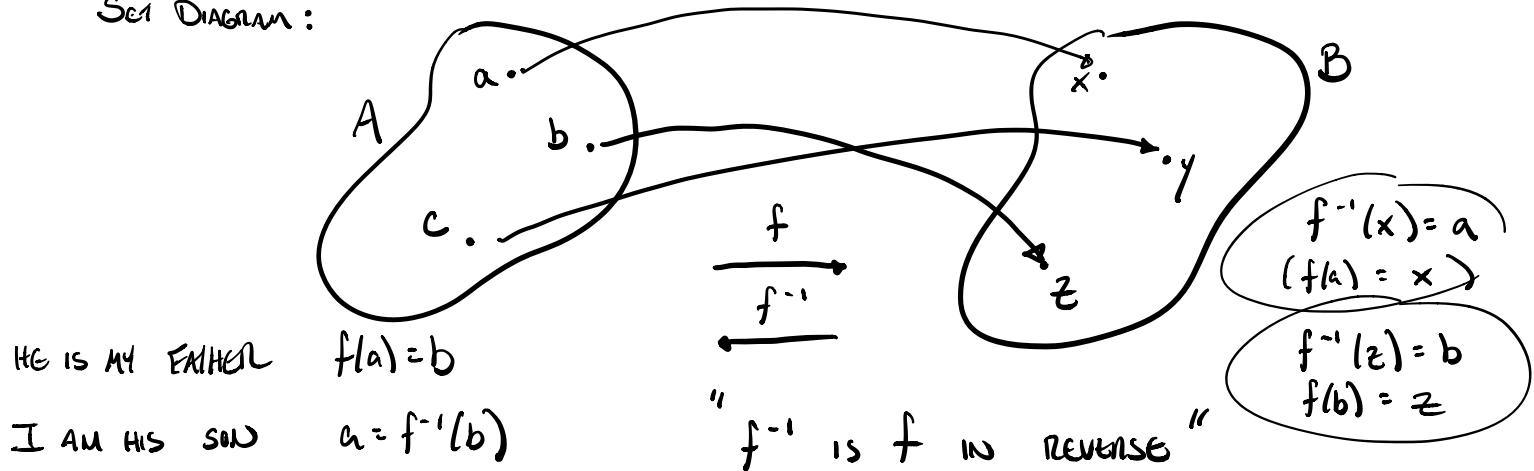
$f(x) = x^3$  is one-to-one.



NOTATIONS:

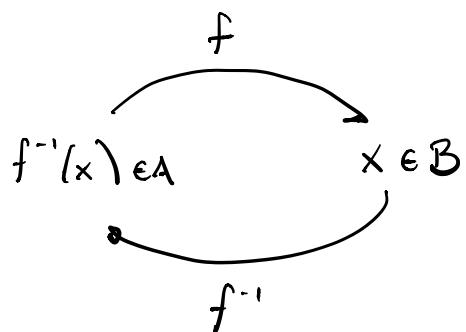
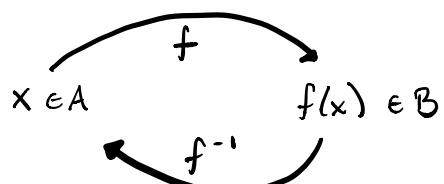
$f^{-1}(x)$  "f-inverse of x"

Set Diagram:



$$f^{-1}(f(x)) = x \quad \text{for every } x \in A$$

$$f(f^{-1}(x)) = x \quad \text{for every } x \in B$$



17. Assume that  $f$  is a one-to-one function.

- (a) If  $f(6) = 17$ , what is  $f^{-1}(17)$ ?
- (b) If  $f^{-1}(3) = 2$ , what is  $f(2)$ ?

$$17 \text{ (a)} \quad f^{-1}(f(6)) = f^{-1}(17)$$

$$f^{-1}(17) = 6$$

18. If  $f(x) = x^5 + x^3 + x$ , find  $f^{-1}(3)$  and  $f(f^{-1}(2))$ .

19. If  $h(x) = x + \sqrt{x}$ , find  $h^{-1}(6)$ .

$$(b) \quad f^{-1}(3) = 2$$

$$f(f^{-1}(3)) = f(2)$$

$$3 = f(2)$$

18. INTRODUCE SYMBOL FOR UNKNOWN:  $f^{-1}(3) = x$

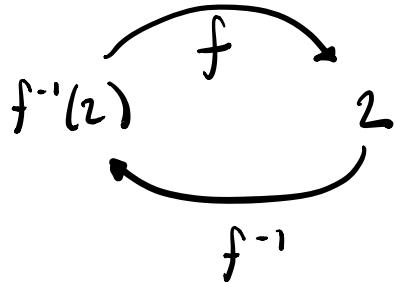
$$f(x) = x^5 + x^3 + x = 3$$

Do we know  $x$ ?  $x = 1$

$$f(1) = 3 \iff f^{-1}(3) = 1$$

$$f(f^{-1}(2)) = 2$$



**5 How to Find the Inverse Function of a One-to-One Function  $f$**

**STEP 1** Write  $y = f(x)$ .

**STEP 2** Solve this equation for  $x$  in terms of  $y$  (if possible).

**STEP 3** To express  $f^{-1}$  as a function of  $x$ , interchange  $x$  and  $y$ .  
The resulting equation is  $y = f^{-1}(x)$ .

**23–28** Find a formula for the inverse of the function.

23.  $f(x) = 5 - 4x$

24.  $f(x) = \frac{4x - 1}{2x + 3}$

25.  $f(x) = 1 + \sqrt{2 + 3x}$

26.  $y = x^2 - x$ ,  $x \geq \frac{1}{2}$

27.  $y = \frac{1 - \sqrt{x}}{1 + \sqrt{x}}$

28.  $f(x) = 2x^2 - 8x$ ,  $x \geq 2$

ex. Let  $f(x) = 1 + \sqrt{2 + 3x}$ . FIND  $f^{-1}(x)$ .

①  $y = 1 + \sqrt{2 + 3x}$

② Solve for  $x$ .

$$y - 1 = \sqrt{2 + 3x}$$

$$(y - 1)^2 = 2 + 3x \quad \rightsquigarrow \quad x = \frac{(y - 1)^2 - 2}{3}$$

③  $y = \frac{(x - 1)^2 - 2}{3} \quad \rightsquigarrow \quad f^{-1}(x) = \frac{(x - 1)^2 - 2}{3}$

Note:  $f(x) = 1 + \sqrt{2 + 3x}$

$$f^{-1}(x) = \frac{(x - 1)^2 - 2}{3}$$

1. MULTIPLY BY 3

2. ADD 2

3. SQRT.

4. ADD 1

1. SUB 1

2. SQUARE

3. SUB 2

4. DIV. BY 3

TIE A KNOT

OPPOSITE OPERATIONS  
IN OPPOSITE ORDER

UNTYE KNOT

The principle of interchanging  $x$  and  $y$  to find the inverse function also gives us the method for obtaining the graph of  $f^{-1}$  from the graph of  $f$ . Since  $f(a) = b$  if and only if  $f^{-1}(b) = a$ , the point  $(a, b)$  is on the graph of  $f$  if and only if the point  $(b, a)$  is on the graph of  $f^{-1}$ . But we get the point  $(b, a)$  from  $(a, b)$  by reflecting about the line  $y = x$ . (See Figure 8.)

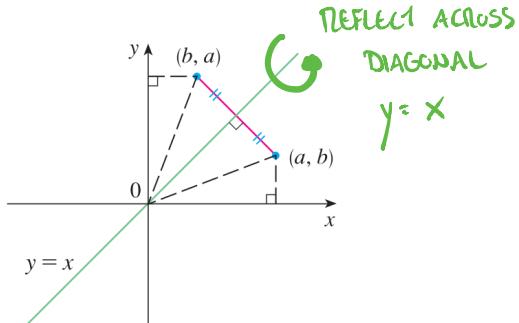


FIGURE 8

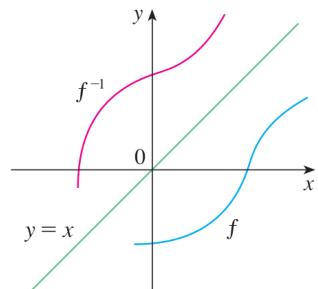


FIGURE 9

Therefore, as illustrated by Figure 9:

$$f(a) = b$$

$(a, b)$  ON GRAPH

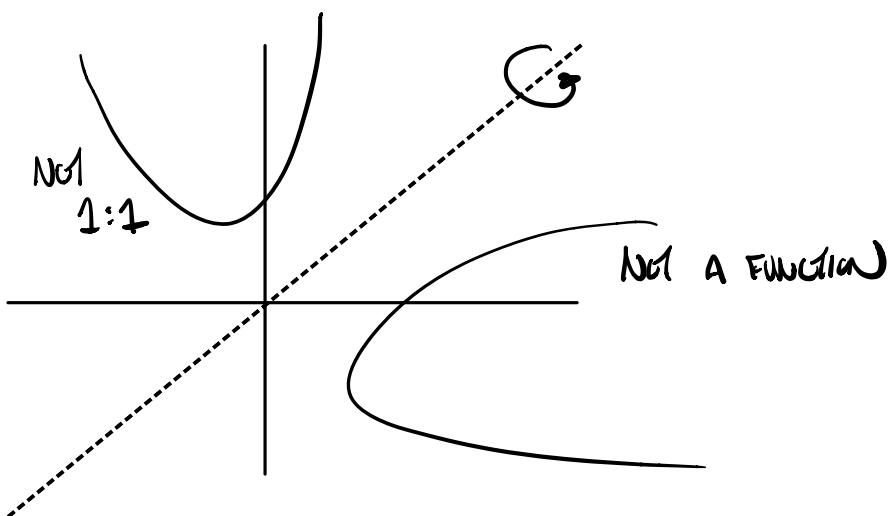
$$y = f(x)$$

$$f^{-1}(b) = a$$

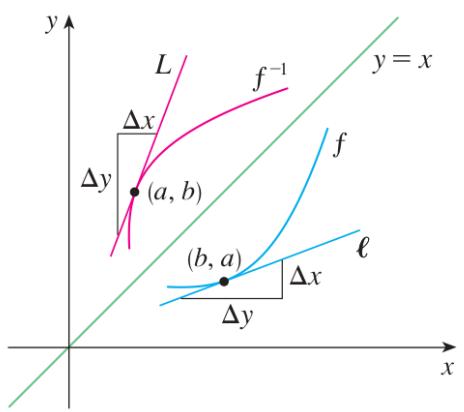
$(b, a)$  ON GRAPH

$$y = f^{-1}(x)$$

The graph of  $f^{-1}$  is obtained by reflecting the graph of  $f$  about the line  $y = x$ .



**6 Theorem** If  $f$  is a one-to-one continuous function defined on an interval, then its inverse function  $f^{-1}$  is also continuous.



**7 Theorem** If  $f$  is a one-to-one differentiable function with inverse function  $f^{-1}$  and  $f'(f^{-1}(a)) \neq 0$ , then the inverse function is differentiable at  $a$  and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

PROOF USING IMPLICIT DIFFERENTIATION:

Let  $y = f^{-1}(x)$ . FIND  $y'$ .

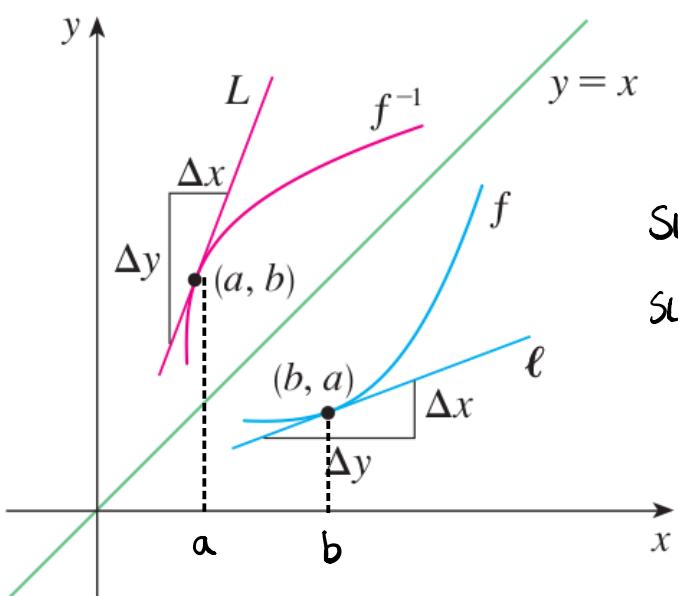


$$f(y) = x \quad \text{IMPLICIT DIFF.}$$

$$\frac{d}{dx} f(y) = \frac{d}{dx} x$$

$$f'(y)y' = 1 \quad (\text{chain rule})$$

$$y' = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))} \quad \checkmark$$



SLOPE OF  $f$  AT  $(b, a)$   
SLOPE OF  $f^{-1}$  AT  $(a, b)$

RECIPROCALS

$$f'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}$$



39-42 Find  $(f^{-1})'(a)$ .

39.  $f(x) = 3x^3 + 4x^2 + 6x + 5, \quad a = 5$

40.  $f(x) = x^3 + 3 \sin x + 2 \cos x, \quad a = 2$

41.  $f(x) = 3 + x^2 + \tan(\pi x/2), \quad -1 < x < 1, \quad a = 3$

42.  $f(x) = \sqrt{x^3 + 4x + 4}, \quad a = 3$

**7 Theorem** If  $f$  is a one-to-one differentiable function with inverse function  $f^{-1}$  and  $f'(f^{-1}(a)) \neq 0$ , then the inverse function is differentiable at  $a$  and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

43. Suppose  $f^{-1}$  is the inverse function of a differentiable function  $f$  and  $f(4) = 5, f'(4) = \frac{2}{3}$ . Find  $(f^{-1})'(5)$ .

44. If  $g$  is an increasing function such that  $g(2) = 8$  and  $g'(2) = 5$ , calculate  $(g^{-1})'(8)$ .

45. If  $f(x) = \int_3^x \sqrt{1+t^3} dt$ , find  $(f^{-1})'(0)$ .

39.  $(f^{-1})'(5) = \frac{1}{f'(f^{-1}(5))} = \frac{1}{f'(0)} = \boxed{\frac{1}{6}}$

$$f'(x) = 9x^2 + 8x + 6$$



$$f^{-1}(5) = 0 \quad \left( \text{b/c } f(0) = 5 \quad \text{GUESS \& CHECK} \right)$$