§ 1.6 Limit Laws

49. Let $g(x) = \dfrac{x^2 + x - 6}{|x - 2|}$.

(a) Find

(i) $\lim_{x \to 2^+} g(x)$ (ii) $\lim_{x \to 2^-} g(x)$

(b) Does $\lim_{x \to 2} g(x)$ exist?

(c) Sketch the graph of g.

$$|x-2| = \begin{cases} x-2 & \text{IF } x-2 \geq 0 \\ & \quad x \geq 2 \\ -(x-2) = 2-x & \text{IF } x-2 < 0 \\ & \quad x < 2 \end{cases}$$

$$g(x) = \begin{cases} \dfrac{(x+3)\cancel{(x-2)}}{\cancel{(x-2)}} = x+3 & \text{IF } x \geq 2 \\[2ex] \dfrac{(x+3)\cancel{(x-2)}}{-\cancel{(x-2)}} = -(x+3) & \text{IF } x < 2 \end{cases}$$

So $\lim_{x \to 2^+} g(x) = \lim_{x \to 2^+} x+3 = 5$

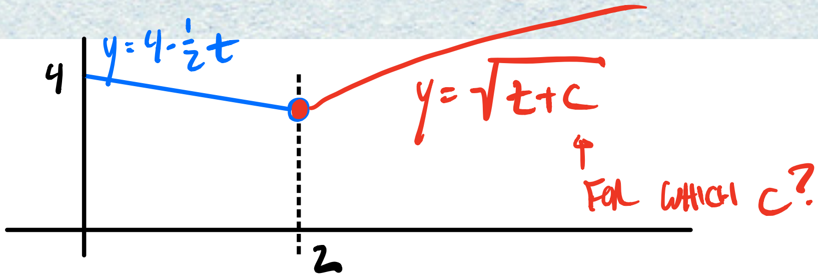
$\lim_{x \to 2^-} g(x) = \lim_{x \to 2^+} -(x+3) = -5$

$\lim_{x \to 2} g(x)$ D.N.E. BECAUSE THESE LIMITS ARE NOT EQUAL!

51. Let

$$B(t) = \begin{cases} 4 - \frac{1}{2}t & \text{if } t < 2 \\ \sqrt{t + c} & \text{if } t \geqslant 2 \end{cases}$$

Find the value of c so that $\lim_{t \to 2} B(t)$ exists.



IN ORDER FOR $\lim_{t \to 2} B(t)$ TO EXIST, WE REQUIRE

$$\lim_{t \to 2^-} B(t) = \lim_{t \to 2^+} B(t)$$

$$\lim_{t \to 2^-} 4 - \frac{1}{2}t = \lim_{t \to 2^+} \sqrt{t + c} \qquad \text{(D.S.R.)}$$

$$3 = \sqrt{2 + c}$$

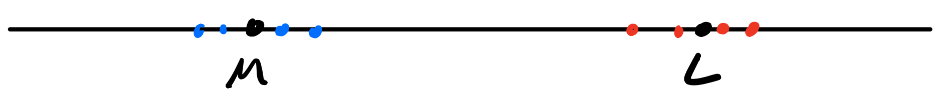
$$9 = 2 + c \implies \boxed{c = 7}$$

<u>THM:</u> IF $f(x) \le g(x)$ FOR x NEAR a, $x \ne a$,

AND IF $\lim_{x \to a} f(x)$ AND $\lim_{x \to a} g(x)$ BOTH EXIST

THEN $\underbrace{\lim_{x \to a} f(x)}_{L} \le \underbrace{\lim_{x \to a} g(x)}_{M}$.

ASSUME IF $L > M$

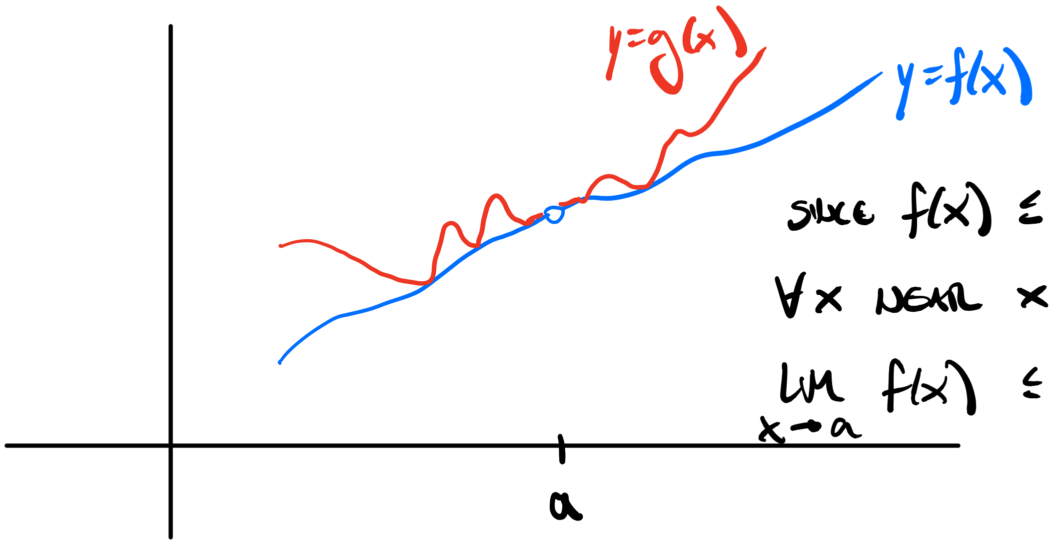


THEN AS $x \to a$, $f(x) \to L$ AND $g(x) \to M$

SO EVENTUALLY, AS $x \to a$, WE MUST HAVE

$$f(x) \ge g(x)$$

[THIS CONTRADICTION IMPLIES THE ASSUMPTION IS WRONG. ✓]



SINCE $f(x) \le g(x)$

$\forall x$ NEAR $x = a$

$\lim_{x \to a} f(x) \le \lim_{x \to a} g(x)$

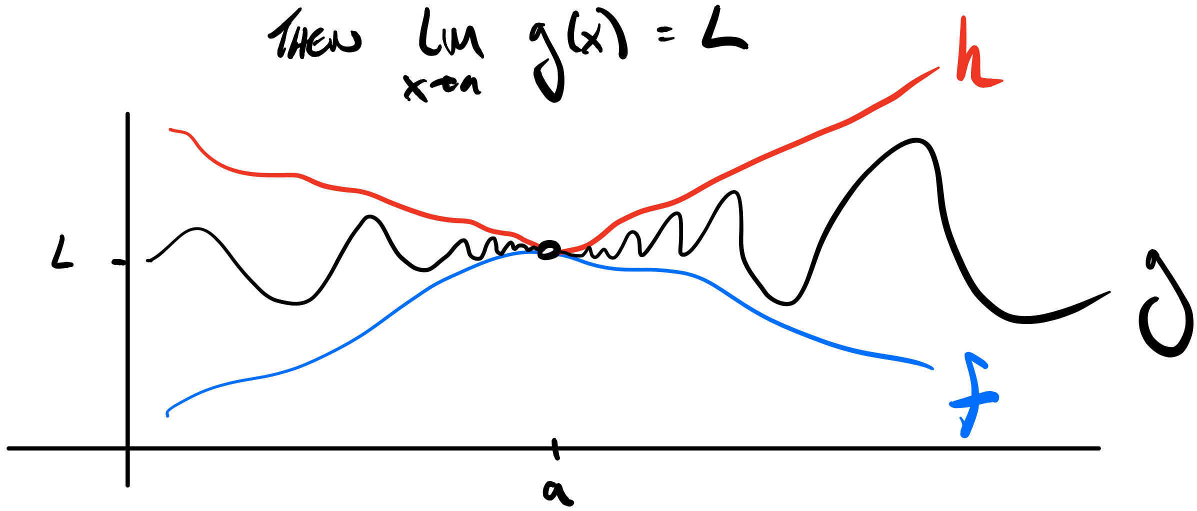
THM (SQUEEZE THM)

IF $f(x) \le g(x) \le h(x)$ FOR x NEAR a, $x \ne a$,

AND IF

$\lim_{x \to a} f(x) = L$ $\qquad$ $\lim_{x \to a} h(x) = L$

THEN $\lim_{x \to a} g(x) = L$



ex. FIND $\lim_{x \to 0} x^4 \cos\left(\frac{2}{x}\right)$.

Note: $-1 \leq \cos(\theta) \leq 1$

You come up with f & h

$$-1 \leq \cos\left(\frac{2}{x}\right) \leq 1, \quad x \neq 0$$

$$\overset{f}{-x^4} \leq x^4\cos\left(\frac{2}{x}\right) \leq \overset{h}{x^4}, \quad x \neq 0$$

Squeeze Thm $\Longrightarrow$

$$\underbrace{\lim_{x\to 0} -x^4}_{0} \leq \lim_{x\to 0} x^4\cos\left(\frac{2}{x}\right) \leq \underbrace{\lim_{x\to 0} x^4}_{0}$$

$$0 \leq \underline{\underline{0}} \leq 0$$

HW #4 §1.6 Due 9/18

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§1.8 Continuity

- Def of continuity
- Types of discontinuities
- Properties: Applying Limit Laws to Continuous Functions
- Compositions of Continuous Functions
- Intermediate Value Thm.

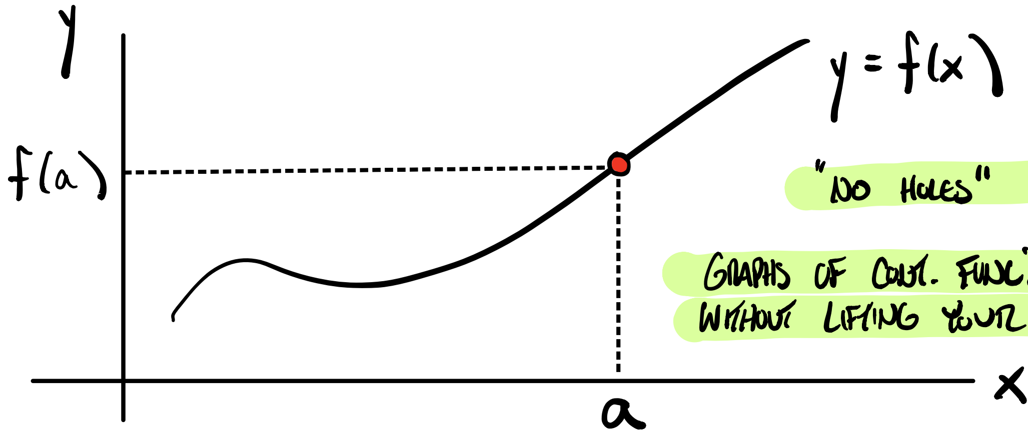
Def: f is continuous at a if

$$\lim_{x\to a} f(x) = f(a)$$

(1) EXISTS

(2) DEFINED

(3) EQUAL



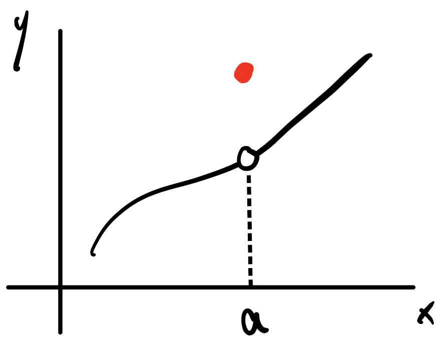
THE GRAPH OF f:

"NO HOLES" "NO BREAKS"

GRAPHS OF CONT. FUNC'S CAN BE DRAWN WITHOUT LIFTING YOUR PENCIL

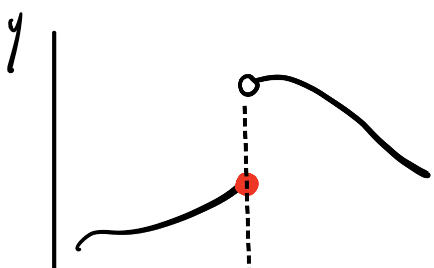
f is **CONTINUOUS ON AN INTERVAL** I IF f IS CONTINUOUS AT EVERY POINT IN I.

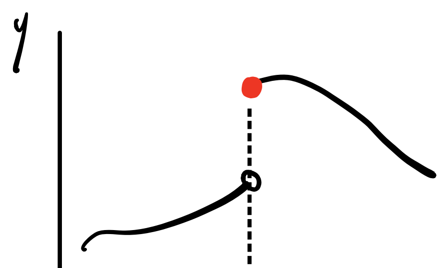
TYPES OF DISCONTINUITIES:

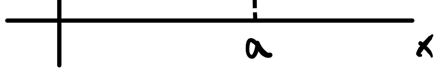


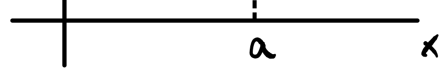
REMOVABLE DISCONTINUITY AT $x = a$

DISCONTINUITIES ARE VALUES OF x WHERE f IS **NOT CONTINUOUS**









JUMP

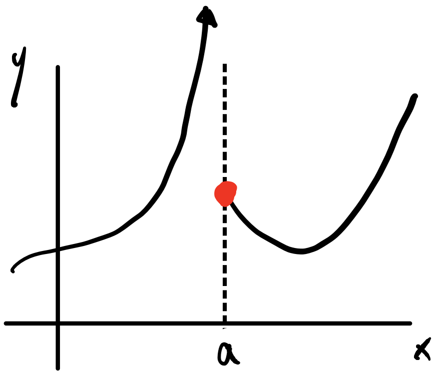
CONTINUOUS FROM THE LEFT

$$\lim_{x \to a^-} f(x) = f(a)$$

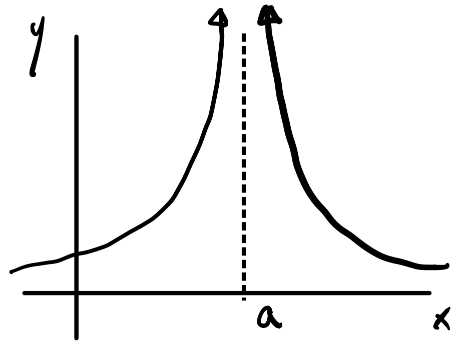
JUMP

CONTINUOUS FROM THE RIGHT

$$\lim_{x \to a^+} f(x) = f(a)$$



INFINITE



INFINITE

AT LEAST ONE OF THE ONE-SIDED LIMITS IS ∞ (D.N.E.)

PROPERTIES OF CONTINUOUS FUNCTIONS:

4 Theorem If f and g are continuous at a and if c is a constant, then the following functions are also continuous at a:

1. $f + g$ **2.** $f - g$ **3.** cf

4. fg **5.** $\dfrac{f}{g}$ if $g(a) \neq 0$

FOLLOWS IMMEDIATELY FROM LIMIT LAWS (§1.6) & DEF. OF CONTINUITY.

e.g. PROOF OF 4. SHOW $\underbrace{\lim_{x \to a} f(x)g(x) = f(a)g(a)}_{fg \text{ is continuous at } x=a}$

$$\lim_{x\to a} f(x)g(x) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$$

$$= f(a) \quad g(a)$$

7 Theorem The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions

BASICALLY A RESTATEMENT OF D.S.R.

LIMITS CAN BE EVALUATED BY PLUGGING IN (ON THEIR DOMAIN)

ex. **39–40** Show that f is continuous on $(-\infty, \infty)$.

39. $f(x) = \begin{cases} 1 - x^2 & \text{if } x \leq 1 \\ \sqrt{x-1} & \text{if } x > 1 \end{cases}$

SHOW $\lim_{x\to a} f(x) = f(a)$ FOR ALL $\underbrace{-\infty < a < \infty}$.

(1) $a < 1$
(2) $a > 1$
(3) $a = 1$

(1) $a < 1$: $\lim_{x\to a} f(x) = \lim_{x\to a} 1 - x^2 = 1 - a^2 = f(a)$

(2) $a > 1$: $\lim_{x\to a} f(x) = \lim_{x\to a} \sqrt{x-1} = \sqrt{\lim_{x\to a} (x-1)}$

$$= \sqrt{a-1} = f(a-1)$$ ✓

(3) SHOW $\lim_{x\to 1} f(x) = f(1)$

$$\left.\begin{array}{l} \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 1-x^2 = 1-(1)^2 = 0 \\[2ex] \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \sqrt{x-1} = \sqrt{1-1} = 0 \end{array}\right\} = f(1)$$ ✓

$$\therefore \lim_{x\to 1} f(x) = f(1)$$

COMPOSITIONS OF CONTINUOUS FUNCTIONS:

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8 Theorem If f is continuous at b and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$

AS $x \to a$, $g(x) \to b$

AS $x \to b$, $f(x) \to f(b)$

$$\underbrace{x}_{\text{CLOSE TO } a} \longmapsto \boxed{g} \longrightarrow \underbrace{g(x)}_{\text{CLOSE TO } b} \longrightarrow \boxed{f} \longrightarrow \underbrace{f(g(x))}_{\text{CLOSE TO } f(b)}$$

LIMITS CAN PASS THROUGH CONTINUOUS FUNCTIONS.

ex. $\lim\limits_{x \to \frac{\pi}{4}} \sin(\sqrt{x})$

SINCE SIN X IS CONTINUOUS ON $(-\infty, \infty)$, THM 8 $\Rightarrow$

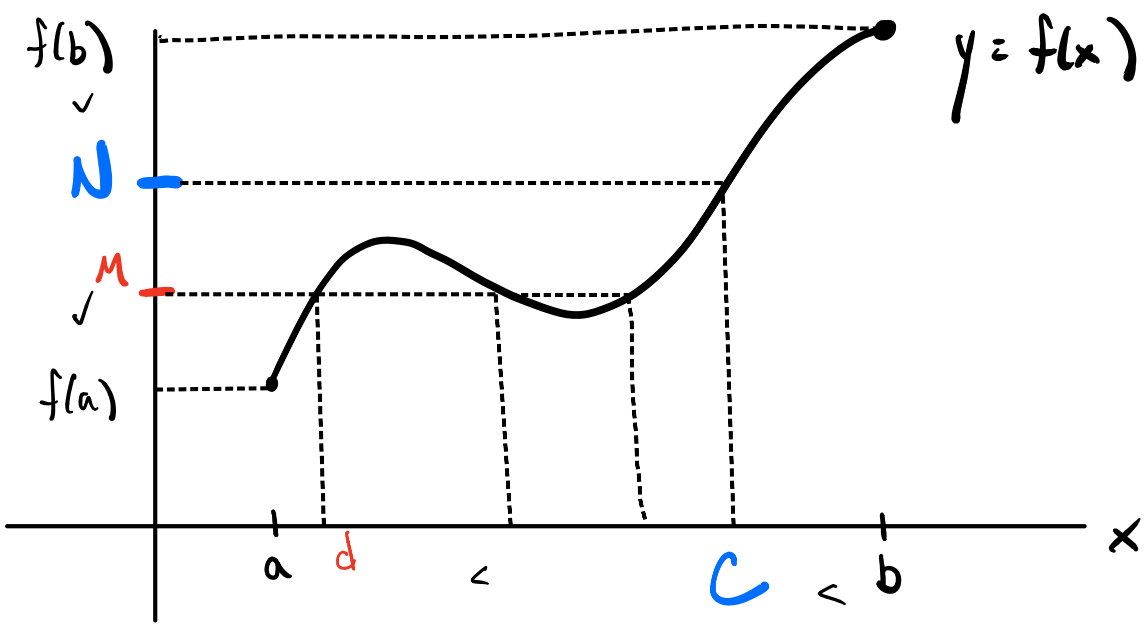
$$= \sin\left(\lim_{x \to \frac{\pi}{4}} \sqrt{x}\right) = \sin\left(\sqrt{\frac{\pi}{4}}\right)$$

9 Theorem If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

i.e. THE COMPOSITION OF CONTINUOUS FUNCTIONS IS CONTINUOUS.

THE INTERMEDIATE VALUE THM:

10 The Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



53–56 Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

53. $x^4 + x - 3 = 0, \quad (1, 2)$

54. $2/x = x - \sqrt{x}, \quad (2, 3)$

55. $\cos x = x, \quad (0, 1)$

56. $\sin x = x^2 - x, \quad (1, 2)$

57–58 (a) Prove that the equation has at least one real root.
(b) Use your calculator to find an interval of length 0.01 that contains a root.

57. $\cos x = x^3$

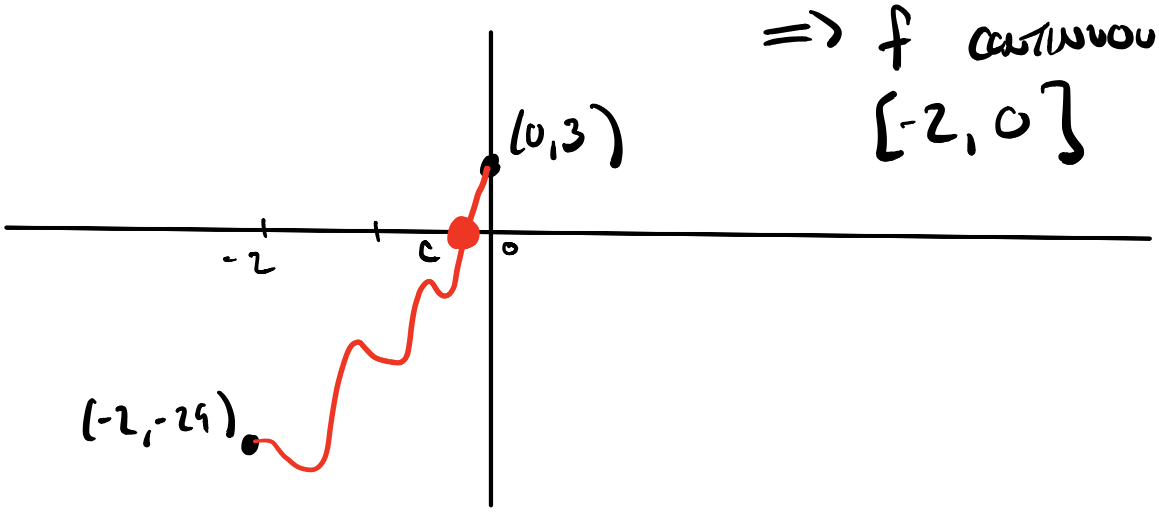
58. $x^5 - x^2 + 2x + 3 = 0$

$f(x) = x^5 - x^2 + 2x + 3$

$f(-2) = -32 - 4 + 4 + 3 = -29 < 0$

$f(0) = 3 > 0$

f IS POLYNOMIAL $\Rightarrow f$ CONTINUOUS ON $[-2, 0]$



SINCE f IS CONT. ON $[-2, 0]$ &
$f(-2) = -29$ & $f(0) = 3$, AND $f(-2) \leq 0 \leq f(0)$

I.V.T. $\Rightarrow$ THERE IS A # $-2 < c < 0$
SUCH THAT $f(c) = 0$