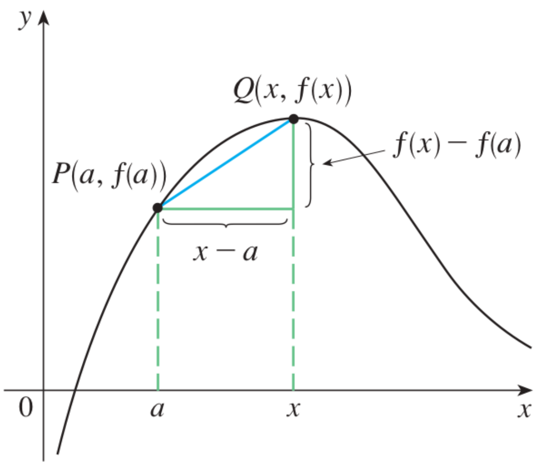
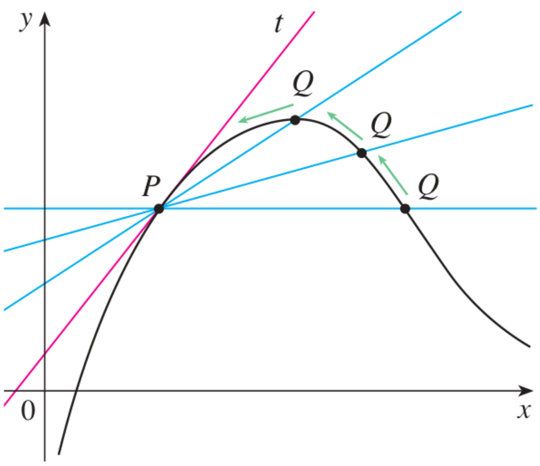
# §2.1 Derivatives & Rates of Change




**Average rate of change** of $f$ between $a$ & $x$ is $\dfrac{f(x)-f(a)}{x-a}$.

Geometrically, this is the slope of the secant line connecting $P(a, f(a))$ & $Q(x, f(x))$.



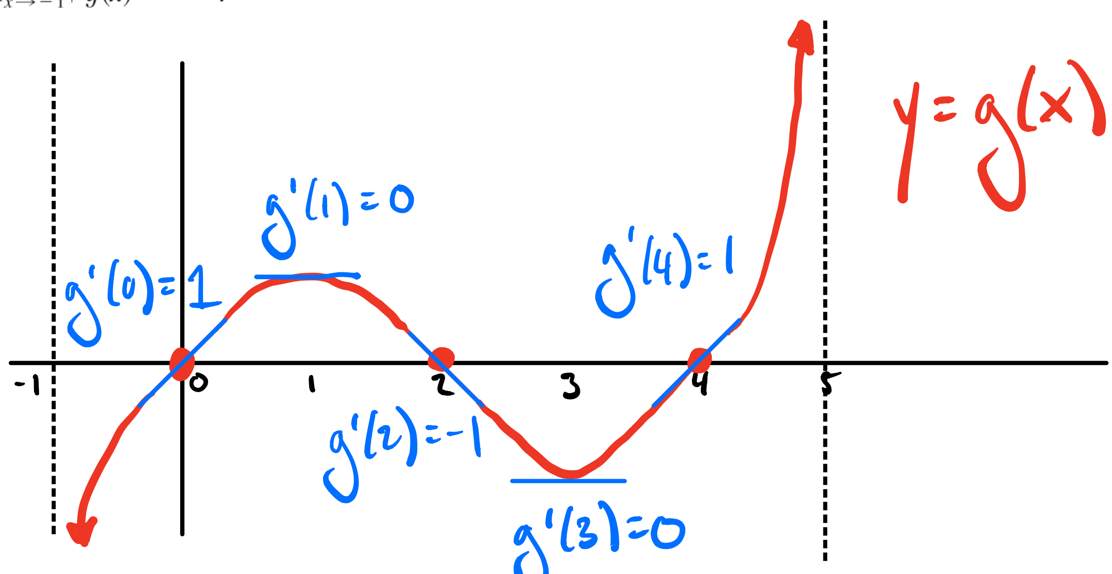

**Instantaneous rate of change** of $f$ at $a$, i.e. the derivative of $f$ at $a$, is

$$\boxed{f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}}.$$

Geometrically, this is the **slope of the tangent line** to the graph $y=f(x)$ at the point $P(a, f(a))$.

"$f$ - prime of $a$"

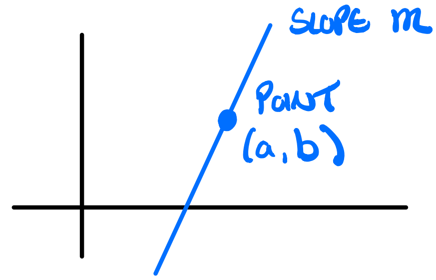
**24.** Sketch the graph of a function $g$ for which $g(0) = g(2) = g(4) = 0$, $g'(1) = g'(3) = 0$, $g'(0) = g'(4) = 1$, $g'(2) = -1$, $\lim_{x\to 5^-} g(x) = \infty$, and $\lim_{x\to -1^+} g(x) = -\infty$.



Thus, the equation of the tangent line to the graph $y = f(x)$ at the point $P(a, f(a))$ is...

Point-slope formula:

$$y - b = m(x - a)$$




$$\dots y - f(a) = f'(a)(x-a)$$

or $$y = f(a) + f'(a)(x-a)$$

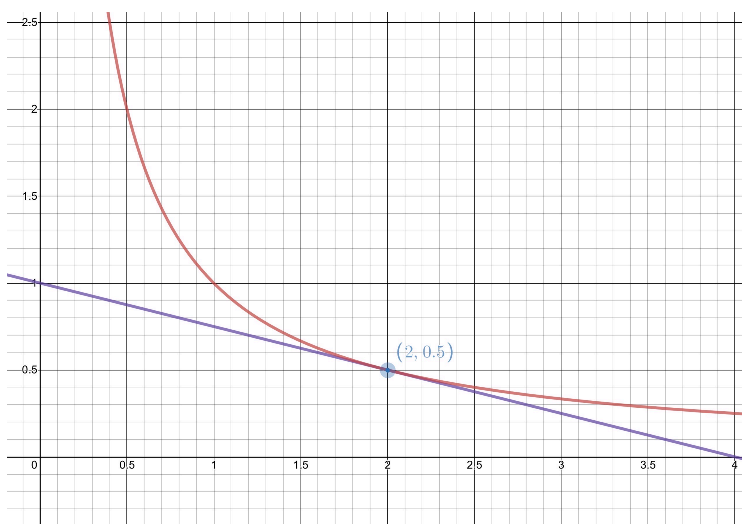
ex. Find the equation of the tangent line to the graph $y = \frac{1}{x}$ at the point $(2, \frac{1}{2})$

Let $f(x) = \frac{1}{x}$, $a = 2$, $f(a) = \frac{1}{2}$

$$y = f(a) + f'(a)(x-a)$$

$$y = \frac{1}{2} + f'(2)(x-2)$$

Find $f'(2)$.




$y = \frac{1}{x}$

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2}$$

$$= \lim_{x \to 2} \frac{2 - x}{2x(x-2)} = \lim_{x \to 2} \frac{-\cancel{(x-2)}}{2x\cancel{(x-2)}}$$

$$= \lim_{x\to 2} -\frac{1}{2x} = -\frac{1}{2(2)} = -\frac{1}{4}$$

$$\boxed{y = \frac{1}{2} + \left(-\frac{1}{4}\right)(x-2) \quad \text{or} \quad y = -\frac{1}{4}x + 1}$$

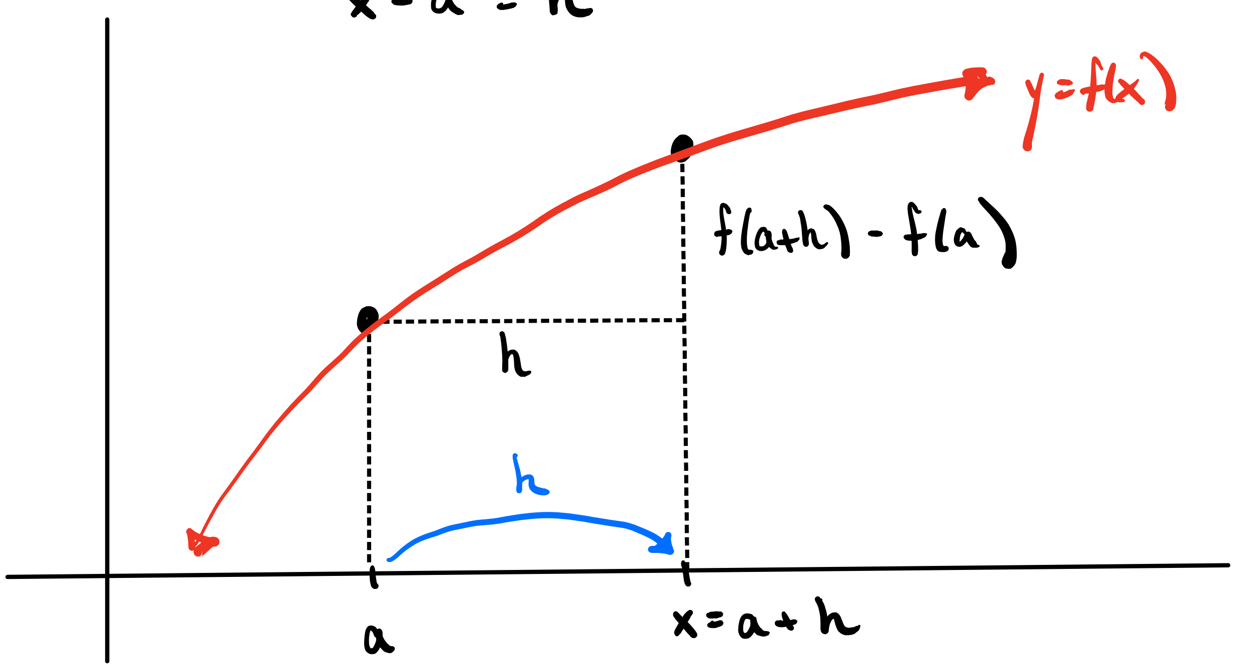
**Equivalent Definition:**

The derivative of $f$ at $a$ is

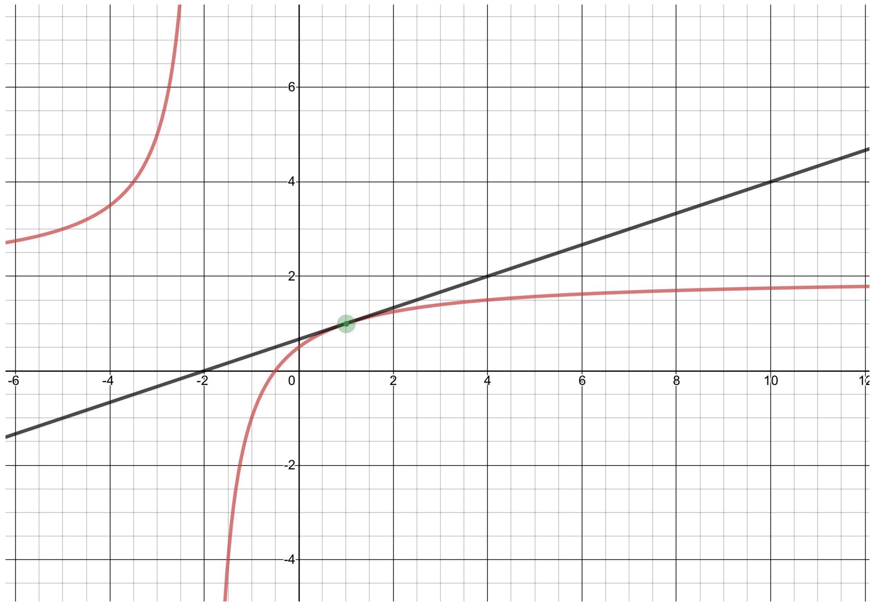
$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Set $x = a+h$     Then $x \to a$ as $h \to 0$

$x - a = h$



EX. FIND AN EQUATION OF THE TANGENT LINE TO THE CURVE $y = \frac{2x+1}{x+2}$ AT THE POINT $(1,1)$.




LINE THRU $(1,1)$

$$y = 1 + m(x-1)$$

WHAT IS THE SLOPE?

$$f'(1)$$

$$f'(1) := \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{2(1+h)+1}{(1+h)+2} - 1}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{3+2h}{3+h} - 1\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{\cancel{3}+2h-(\cancel{3}+h)}{3+h}\right)$$

$$= \lim_{h \to 0} \frac{1}{\cancel{h}} \left( \frac{\cancel{h}}{3+h} \right)$$

$$= \lim_{h \to 0} \frac{1}{3+h} = \frac{1}{3}$$ SLOPE OF TANGENT LINE TO $y = f(x)$ AT $(1,1)$.

$$\boxed{y = 1 + \frac{1}{3}(x-1) \quad \text{or} \quad y = \frac{1}{3}x + \frac{2}{3}}$$

## DERIVATIVES : POSITION & VELOCITY

IF $s(t)$ GIVES THE POSITION OF AN OBJECT MOVING IN A STRAIGHT LINE (1 DIMENSION) AT TIME $t$.

THEN $\frac{s(t) - s(a)}{t - a}$ IS AVERAGE RATE OF CHANGE IN POSITION BETWEEN TIMES $a$ & $t$.

i.e. AVERAGE VELOCITY OF THE OBJECT BETWEEN TIMES $t$ & $a$.

( UNITS $\frac{\text{DISPLACEMENT}}{\text{TIME}}$ = VELOCITY )

THEN THE DERIVATIVE OF S AT a IS

$$s'(a) := \lim_{t \to a} \frac{s(t) - s(a)}{t - a}$$

$$:= \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}$$

GIVES THE INSTANTANEOUS VELOCITY OF THE OBJECT AT TIME a.

$s(a)$ = POSITION AT TIME a

$s'(a)$ = VELOCITY AT TIME a.

**14.** If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height (in meters) after $t$ seconds is given by $H = 10t - 1.86t^2$.

(a) Find the velocity of the rock after one second.

(b) Find the velocity of the rock when $t = a$.

(c) When will the rock hit the surface?

(d) With what velocity will the rock hit the surface?

(b.) VELOCITY OF THE ROCK WHEN $t = a$

$$= H'(a) := \lim_{h \to 0} \frac{H(a+h) - H(a)}{h}$$

( THE DERIVATIVE OF H AT a. )

$$= \lim_{h \to 0} \frac{10(a+h) - 1.86(a+h)^2 - [10a - 1.86a^2]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{10a} + 10h - 1.86(\cancel{a^2} + 2ah + h^2) - \cancel{10a} + \cancel{1.86a^2}}{h}$$

$$= \lim_{h \to 0} \frac{10h - 3.72ah - 1.86h^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(10 - 3.72a - 1.86h)}{\cancel{h}}$$

$$= \lim_{h \to 0} 10 - 3.72a - 1.86h$$

$$= \boxed{10 - 3.72a}$$

VELOCITY OF THE ROCK AT TIME $a$.

(a) VELOCITY AT TIME 1

$$= H'(1) = 10 - 3.72(1) = \boxed{6.28 \text{ m/s}}$$

$$\underbrace{H_1 = 0}$$

(c.) WHEN WILL THE ROCK HIT THE SURFACE?

$$H(t) = 10t - 1.86t^2 = 0$$

$$t(10 - 1.86t) = 0$$

$t = 0$ (THROWN)

$$\boxed{t = \frac{10}{1.86}}$$

(HITS THE SURFACE)

(d.) VELOCITY OF THE ROCK WHEN IT HITS THE SURFACE.

$$H'\left(\frac{10}{1.86}\right) = 10 - 3.72\left(\frac{10}{1.86}\right)$$

$$= \boxed{-10 \text{ m/s}}$$

HITS THE GROUND AT SAME SPEED AT WHICH IT WAS THROWN.

**51.** The cost (in dollars) of producing $x$ units of a certain commodity is $C(x) = 5000 + 10x + 0.05x^2$.

(a) Find the average rate of change of $C$ with respect to $x$ when the production level is changed

(i) from $x = 100$ to $x = 105$

(ii) from $x = 100$ to $x = 101$

(b) Find the instantaneous rate of change of $C$ with respect to $x$ when $x = 100$. (This is called the *marginal cost*. Its significance will be explained in Section 2.7.)

(a.) (i.) FROM $x = 100$ TO $x = 101$

$$C(101) - C(100)$$

$$= 5000 + 10(101) + .05(101)^2 - 5000 - 10(100) - .05(100)^2$$

$$= 10 + .05(201) = 20.05$$

INTERPRETATION: COST OF PRODUCING $101^{ST}$ UNIT IS \$20.05

(b.) $$C'(100) := \lim_{h \to 0} \frac{C(100+h) - C(100)}{h}$$

$$= \lim_{h \to 0} \frac{5000 + 10(100+h) + .05(100+h)^2 - 5000 - 10(100) - .05(100)^2}{h}$$

$$= \ldots = 10 + .1(100) = 20$$

$C'(100)$ IS CALLED THE MARGINAL COST WHEN PRODUCING 100 ITEMS.

IT IS APPROXIMATELY THE COST OF PRODUCING ONE ADDITIONAL ITEM, WHEN PRODUCING 100 ITEMS.