§2.3 DIFFERENTIATION RULES

$\frac{d}{dx} c = 0$, C IS ANY CONSTANT

$\frac{d}{dx} x^n = nx^{n-1}$, n IS ANY POSITIVE INTEGER.

LET f & g BE DIFFERENTIABLE FUNCTIONS. a, b, c ARE CONSTANTS

$\frac{d}{dx} cf(x) = c\frac{d}{dx} f(x) = cf'(x)$ CONSTANT MULTIPLE RULE

$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x) = f'(x) \pm g'(x)$

SUM & DIFFERENCE RULE.

PROOF: $\frac{d}{dx}\left[af(x) \pm bg(x)\right]$

$$= \lim_{h\to 0} \frac{\left(af(x+h) \pm bg(x+h)\right) - \left(af(x) \pm bg(x)\right)}{h}$$

$$= \lim_{h\to 0} \left[a\frac{f(x+h)-f(x)}{h} \pm b\frac{g(x+h)-g(x)}{h}\right]$$

$$= a\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right) \pm b\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right)$$

$$= af'(x) \pm bg'(x) \qquad \square$$

NOW WE CAN DIFFERENTIATE ANY POLYNOMIAL.

ex. Let $f(x) = x^4 + 2x^3 - \frac{7}{2}x^2 + \sqrt{2}\,x + \pi^3$

FIND $f'(x)$, $f''(x)$, $f'''(x)$, $f^{(4)}(x)$, $f^{(5)}(x)$, ETC...

$$\frac{d}{dx}\left[x^4 + 2x^3 - \frac{7}{2}x^2 + \sqrt{2}\,x + \pi^3\right]$$

$$= \frac{d}{dx}\left[x^4\right] + \frac{d}{dx}\left[2x^3\right] - \frac{d}{dx}\left[\frac{7}{2}x^2\right] + \frac{d}{dx}\left[\sqrt{2}\,x\right] + \frac{d}{dx}\left[\pi^3\right]$$

$$= \frac{d}{dx}\left[x^4\right] + 2\frac{d}{dx}\left[x^3\right] - \frac{7}{2}\left[\frac{d}{dx}x^2\right] + \sqrt{2}\frac{d}{dx}\left[x^1\right] + \frac{d}{dx}\left[\pi^3\right]$$

$$= 4x^{4-1} + 2\cdot 3x^{3-1} + \frac{7}{2}\cdot 2x^{2-1} + \sqrt{2}\cdot 1\underbrace{x^{1-1}}_{1} + 0$$

$$f'(x) = 4x^3 + 6x^2 + 7x + \sqrt{2}$$

$$f''(x) = \frac{d}{dx}\left[4x^3 + 6x^2 + 7x + \sqrt{2}\right]$$

$$= 4\cdot 3x^2 + 6\cdot 2x + 7$$

$$f''(x) = 12x^2 + 12x + 7$$

$$f'''(x) = 12\cdot 2x + 12 = 24x + 12$$

$$f^{(4)}(x) = 24$$

$$f^{(5)}(x) = 0$$

$$f^{(6)}(x) = 0$$

$\vdots \qquad \vdots$

IF $P(x)$ IS AN n^{TH} DEGREE POLYNOMIAL

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$$

THEN $\frac{d^n}{dx^n}P(x) = \text{const.}$ & $\frac{d^m}{dx^m}P(x) = 0$

FOR $m > n$.

PRODUCT RULE: $\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x)$

PROOF: $\frac{d}{dx}\left[f(x)g(x)\right] = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\left[g(x+h)\frac{f(x+h)-f(x)}{h} + f(x)\frac{g(x+h)-g(x)}{h}\right]$$

$$= \lim_{h\to 0} g(x+h)\cdot \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

g IS DIFFERENTIABLE $\Rightarrow$ g IS CONTINUOUS.

$$g(x)f'(x) + f(x)g'(x) \quad \square$$

ex. LET $f(x) = (x^2 - x)(3x+1)$

FIND $f'(x)$ WITH & WITHOUT USING THE PRODUCT RULE.

$$\frac{d}{dx}\Big[\underbrace{(x^2-x)}_{f}\underbrace{(3x+1)}_{g}\Big] =$$

$$= \underbrace{\frac{d}{dx}\left[x^2-x\right]}_{f'}\underbrace{(3x+1)}_{g} + \underbrace{(x^2-x)}_{f}\underbrace{\frac{d}{dx}\left[3x+1\right]}_{g'}$$

$$= (2x-1)(3x+1) + (x^2-x)(3)$$

$= 6x^2 + 2x - 3x - 1 + 3x^2 - 3x$

$= \boxed{9x^2 - 4x - 1}$

$f(x) = (x^2 - x)(3x+1) = 3x^3 + x^2 - 3x^2 - x$

$= 3x^3 - 2x^2 - x$

$f'(x) = 3 \cdot 3x^2 - 2 \cdot 2x - 1$

$= 9x^2 - 4x - 1$ ✓

QUOTIENT RULE : $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

$f(x) \sim$ HI

$g(x) \sim$ LO

$\dfrac{d}{dx}\left[\dfrac{HI}{LO}\right] = \dfrac{LO \cdot DHI - HI\ DLO}{LO^2}$

PROOF : $\lim\limits_{h \to 0} \dfrac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h}$

NOTE: WE ASSUME $g(x) \neq 0$

$= \lim\limits_{h \to 0} \dfrac{1}{h}\left[\dfrac{g(x)f(x+h) - f(x)g(x+h)}{g(x+h)g(x)}\right]$

$= \lim\limits_{h \to 0} \dfrac{1}{h}\left[\dfrac{g(x)f(x+h) - g(x)f(x) - (f(x)g(x+h) - f(x)g(x))}{g(x+h)g(x)}\right]$

$= \lim\limits_{h \to 0} \dfrac{1}{g(x+h)g(x)}\left(g(x)\dfrac{f(x+h)-f(x)}{h} - f(x)\dfrac{g(x+h)-g(x)}{h}\right)$

$$= \lim_{h\to 0} \frac{1}{g(x+h)g(x)} \left[g(x) \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} - f(x) \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \right]$$

$$= \frac{1}{(g(x))^2} \left(g(x) f'(x) - f(x) g'(x) \right) \quad \square$$

ex. FIND $\frac{d}{dx}\left[\frac{x^2+1}{2x^3-4x+5}\right]$

LET $f(x) = x^2+1$ $\qquad f'(x) = 2x$

$g(x) = 2x^3 - 4x + 5$ $\qquad g'(x) = 2 \cdot 3x^2 - 4$

$= 6x^2 - 4$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

$$\boxed{= \frac{(2x^3-4x+5)(2x) - (x^2+1)(6x^2-4)}{(2x^3-4x+5)^2}}$$

ex. FIND $\frac{d}{dx}\left[\frac{1}{x^n}\right]$, $n = 1, 2, 3, \ldots$

QUOTIENT RULE:

$$\frac{x^n \cdot \overbrace{\frac{d}{dx}[1]}^{0} - 1 \cdot \frac{d}{dx}[x^n]}{(x^n)^2}$$

$$= \frac{-1 \cdot n x^{n-1}}{x^{2n}} = -n x^{n-1-2n}$$

$$\frac{d}{dx}\left[x^{-n}\right] \longrightarrow = -n x^{-n-1}$$

$\therefore$ POWER RULE WORKS FOR NEGATIVE INTEGER EXPONENTS AS WELL!

GENERAL POWER RULE: $\frac{d}{dx}\left[x^{n}\right] = n x^{n-1}$

FOR ANY REAL NUMBER n

ex. $$\frac{d}{dx} x^{2/3} = \frac{2}{3} x^{2/3 - 1} = \frac{2}{3} x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

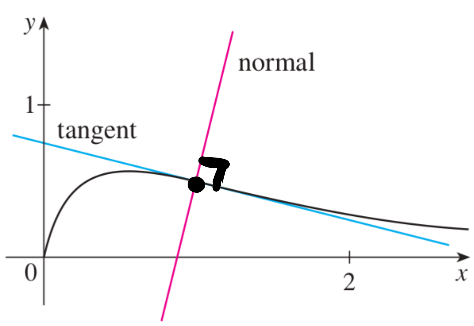
$$\left(x^{-n} = \frac{1}{x^n} \quad \& \quad x^{1/n} = \sqrt[n]{x} \right)$$

ex. $$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}\left[x^{1/2}\right] = \frac{1}{2} x^{1/2 - 1} = \frac{1}{2} x^{-1/2}$$

$$= \frac{1}{2\sqrt{x}}$$

ex.

EXAMPLE 12 Find equations of the tangent line and normal line to the curve $y = \sqrt{x}/(1 + x^2)$ at the point $\left(1, \frac{1}{2}\right)$.



NORMAL LINE $\perp$ TANGENT LINE

SLOPES OF THESE LINES ARE

NEG. RECIP. OF EACHOTHER.

$$y = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{1+x^2} \qquad f'(x) = \frac{1}{2\sqrt{x}}$$

$$g'(x) = 2x$$

$$y' = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} = \frac{(1+x^2)\frac{1}{2\sqrt{x}} - \sqrt{x}\cdot 2x}{(1+x^2)^2}$$

AT $x=1$: $$y' = \frac{(1+1)\frac{1}{2} - 1\cdot 2}{(1+1)^2} = \frac{1-2}{4} = -\frac{1}{4}$$

POINT: $\left(1, \frac{1}{2}\right)$, SLOPE: $-\frac{1}{4}$

$$y - \frac{1}{2} = -\frac{1}{4}(x-1) \qquad \text{TANGENT LINE}$$

$$y - \frac{1}{2} = 4(x+1) \qquad \text{NORMAL LINE}$$

(SLOPES ARE NEG. RECIP.)