ex. $y = x^2 \sin x \tan x$. FIND y'.

$$y = \underbrace{x^2}_{f(x)} \underbrace{\sin x \tan x}_{g(x)}$$

$f'(x) = 2x$

$g(x) = \sin x \tan x$

$g'(x) = \sin x \frac{d}{dx}[\tan x] + \frac{d}{dx}[\sin x] \tan x$

$= \sin x \sec^2 x + \cos x \tan x$

$$\underbrace{y = f(x)\, g'(x) + f'(x)\, g(x)}$$

$$y' = x^2(\sin x \sec^2 x + \cos x \tan x) + 2x \sin x \tan x$$

$$= x^2 \sin x \sec^2 x + x^2 \cos x \tan x + 2x \sin x \tan x$$

$$y = x^2 \sin x \tan x$$

$$y' = \frac{d}{dx}[x^2] \sin x \tan x + x^2 \frac{d}{dx}[\sin x] \tan x + x^2 \sin x \frac{d}{dx}[\tan x]$$

ex. $y = \dfrac{\sqrt{x} \sin x}{1 + \tan x}$. FIND y'

IF $y = \dfrac{f(x)}{g(x)}$

$$y' = \frac{g(x) f'(x) - f(x) g'(x)}{g(x)^2}$$

$x^{1/2} \sin x$

$$y' = \frac{(1 + \tan x) \frac{d}{dx}[\sqrt{x} \sin x] - \sqrt{x} \sin x \frac{d}{dx}[1 + \tan x]}{(1 + \tan x)^2}$$

$$y' = \frac{(1 + \tan x)\left[\frac{1}{2} x^{-1/2} \sin x + x^{1/2} \cos x\right] - \sqrt{x} \sin x [\sec^2 x]}{(1 + \tan x)^2}$$

ex. FIND $\lim_{x \to 0} \frac{\sin(5x)}{3x}$.

$$= \lim_{x \to 0} \frac{\sin(5x)}{5x} \cdot \frac{5}{3}$$

$$= \frac{5}{3} \lim_{x \to 0} \frac{\sin(5x)}{5x}$$

LET $u = 5x$ ← CONTINUOUS FUNCTION.

$$\lim_{x \to 0} u = \lim_{x \to 0} 5x = 0$$

AS $x \to 0$, $u \to 0$

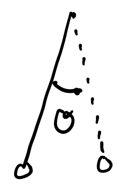
$$\frac{5}{3} \lim_{u \to 0} \frac{\sin(u)}{u} = \frac{5}{3}(1) = \boxed{\frac{5}{3}}$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

ENGINEERING

$\frac{\sin x}{x} \approx 1$ WHEN $|x|$ IS SMALL

$\sin x \approx x$ WHEN $|x|$ IS SMALL



--- $\frac{\sin \theta}{\theta}$...

$$\lim_{x \to 0} \frac{\sin(ax)}{bx} = \frac{a}{b}$$

ex. FIND $\lim_{\theta \to 0} \frac{\cos\theta - 1}{2\theta^2}$.

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{2\theta^2} \cdot \frac{\cos\theta + 1}{\cos\theta + 1}$$

$\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta - 1 = -\sin^2\theta$

$$= \lim_{\theta \to 0} \frac{\cos^2\theta - 1}{2\theta^2(\cos\theta + 1)} = \lim_{\theta \to 0} \frac{-\sin^2\theta}{2\theta^2(\cos\theta + 1)}$$

$$= \lim_{\theta \to 0} \left[-\left(\frac{\sin\theta}{\theta}\right)^2 \left(\frac{1}{2(\cos\theta + 1)}\right) \right]$$

$$= -1\left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right)^2\left(\lim_{\theta\to 0}\frac{1}{2(\cos\theta+1)}\right)$$

$$= -1(1)^2\left(\frac{1}{4}\right) = \boxed{-\frac{1}{4}}$$

§2.5 Chain Rule

Chain Rule: Suppose $f(x)$ & $g(x)$ are differentiable functions.

If we define $F(x) = f(g(x)) = f\circ g\,(x)$

Then the derivative of $F(x)$ is

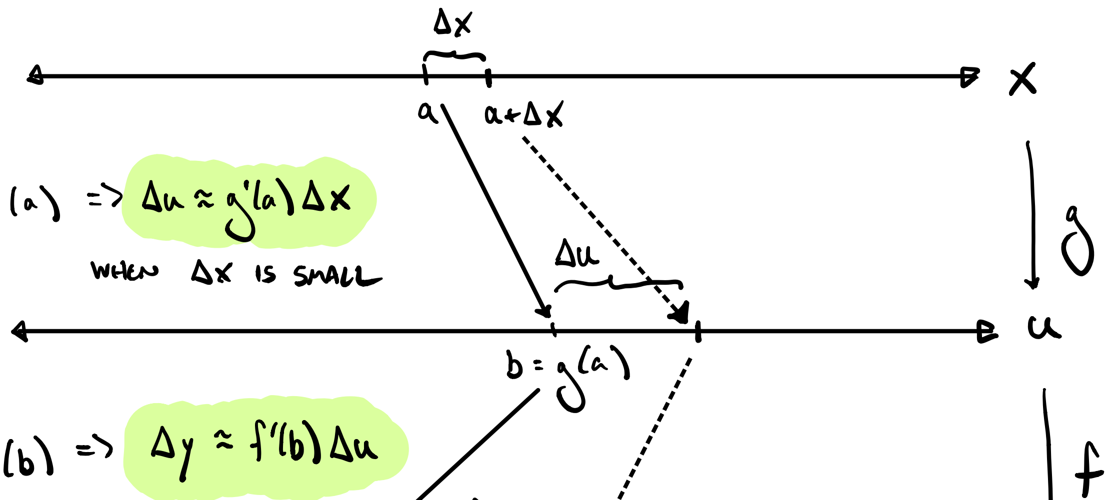
$$F'(x) = f'(g(x))\,g'(x)$$

In other words, if $y = f(u)$, $u = g(x)$

$$x \xrightarrow{g} g(x) = u \xrightarrow{f} f(u) \qquad = f(g(x))$$

Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx}$

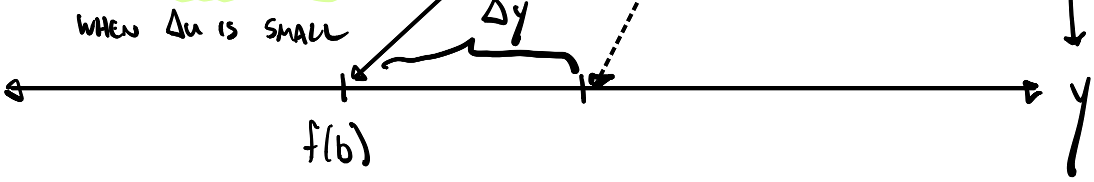
IDEA



$\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x} = g'(a) \Rightarrow \Delta u \approx g'(a)\,\Delta x$

when Δx is small

$\lim_{\Delta u\to 0}\frac{\Delta y}{\Delta u} = f'(b) \Rightarrow \Delta y \approx f'(b)\,\Delta u$



$$\Longrightarrow \quad \Delta y \approx f'(b)\Delta u \approx f'(b)\, g'(a)\, \Delta x$$

$$\frac{\Delta y}{\Delta x} \approx f'(b)\, g'(a) \qquad \text{WHEN } \Delta x \text{ IS SMALL}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(b)\, g'(a) = f'(g(a))\, g'(a)$$

---

ex. $y = \sin(3x^2 + x) = \sin(3x^2 + x)$

$f(x) = \sin x \qquad g(x) = 3x^2 + x$

$y = f(g(x)) \qquad f'(x) = \cos x \qquad g'(x) = 6x + 1$

$y' = f'(g(x))\, g'(x)$

$y' = \cos(g(x))\,(6x+1)$

$$\boxed{y' = \cos(3x^2 + x)(6x+1)}$$

ex. $y = \left(\dfrac{\sin x + 2}{x^2 + 1}\right)^{33} = \left(\dfrac{\sin x + 2}{x^2 + 1}\right)^{33}$

$$y = f(u) = u^{33}, \quad u = g(x) = \frac{\sin x + 2}{x^2+1}$$

$$f'(u) = 33u^{32}$$

$$g'(x) = \frac{(x^2+1)\frac{d}{dx}[\sin x + 2] - (\sin x + 2)\frac{d}{dx}[x^2+1]}{(x^2+1)^2}$$

$$g'(x) = \frac{(x^2+1)\cos x - (\sin x + 2)(2x)}{(x^2+1)^2}$$

$$y' = f'(u)\,g'(x) = 33u^{32}\left(\frac{(x^2+1)\cos x - (\sin x + 2)(2x)}{(x^2+1)^2}\right)$$

$$\boxed{y' = 33\left(\frac{\sin x + 2}{x^2+1}\right)^{32}\left(\frac{(x^2+1)\cos x - (\sin x + 2)(2x)}{(x^2+1)^2}\right)}$$

ex. ex. $y = \sqrt{4x^3 + \sec x} = f(g(x))$

$f(x) = \sqrt{x} = x^{1/2}$

$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$g(x) = 4x^3 + \sec x$

$g'(x) = 12x^2 + \sec x \tan x$

$$y' = f'(g(x))\,g'(x) = \frac{1}{2\sqrt{g(x)}} \cdot (12x^2 + \sec x \tan x)$$

$$\boxed{\begin{aligned} y' &= \frac{12x^2 + \sec x \tan x}{2\sqrt{4x^3 + \sec x}} \\ &= \frac{1}{2}(4x^3 + \sec x)^{-1/2}(12x^2 + \sec x \tan x) \end{aligned}}$$

CHAIN RULE WITH POWER RULE:

$$\frac{d}{dx}\left[f(x)^n \right] = n\, f(x)^{n-1} f'(x)$$

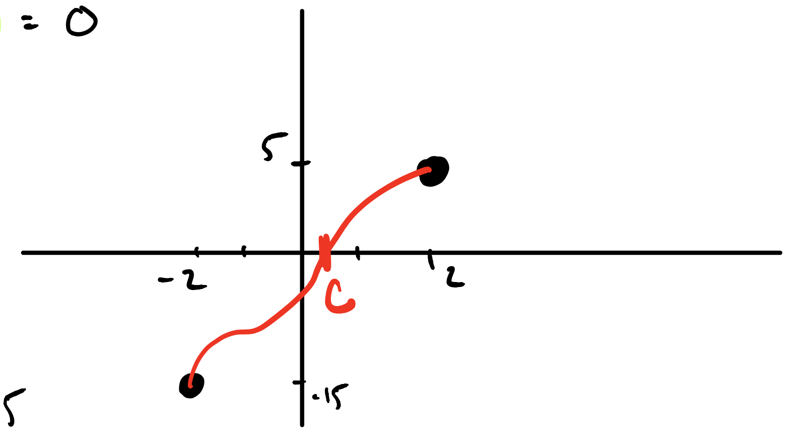
SHOW THIS HAS AT LEAST ONE SOLUTION.

$$x^3 + x = x^2 + 1 \longrightarrow f(x) = 0$$

$$f(x) = x^3 - x^2 + x - 1 = 0$$

f IS POLYNOMIAL

$\Rightarrow f$ IS CONTINUOUS ON $(-\infty, \infty)$



$$f(-2) = -8 - 4 - 2 - 1 = -15$$

$$f(2) = 8 - 4 + 2 - 1 = 5$$

SINCE $f(-2) = -15$ & $f(2) = 5$

AND f IS CONTINUOUS ON $[-2, 2]$

BY THE INTERMEDIATE VALUE THEOREM,

THERE EXISTS A NUMBER c, $-2 < c < 2$,

SUCH THAT $f(c) = 0$

i.e. $c^3 - c^2 + c - 1 = 0$

i.e. $c^3 + c = c^2 + 1$