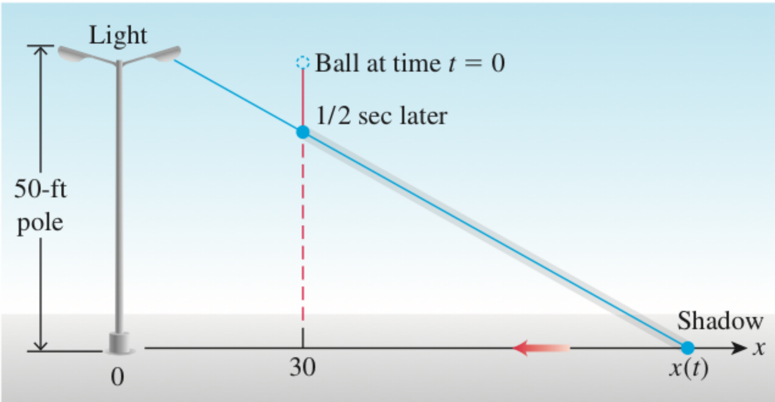
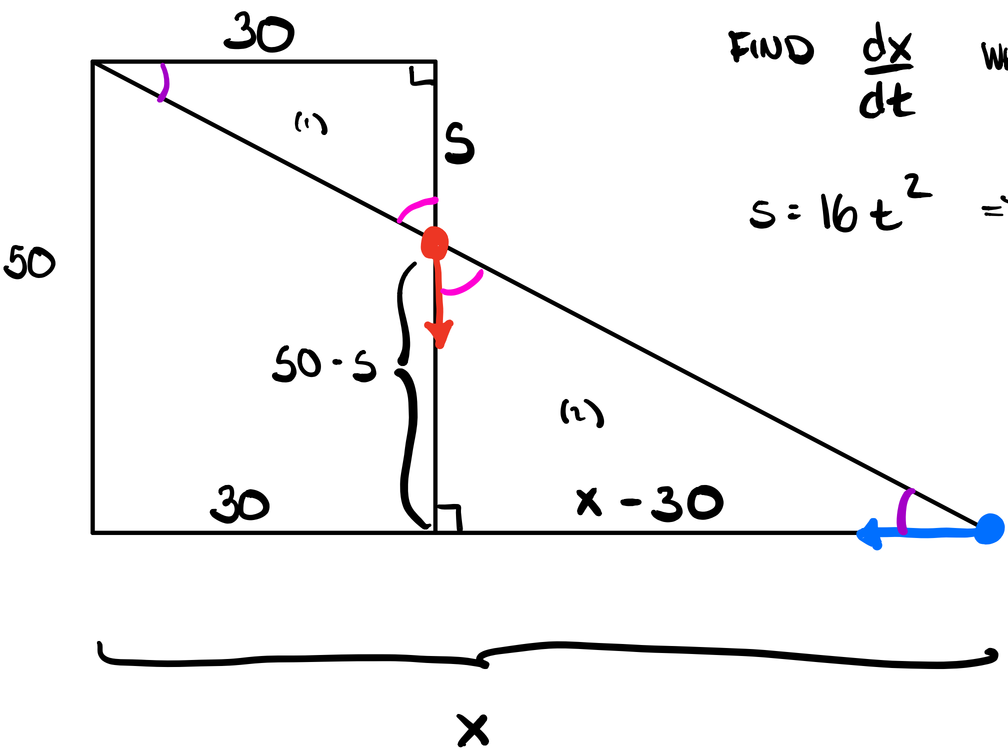
A moving shadow A light shines from the top of a pole 50 ft high. A ball is dropped from the same height from a point 30 ft away from the light. (See accompanying figure.) How fast is the shadow of the ball moving along the ground 1/2 sec later? (Assume the ball falls a distance $s = 16t^2$ ft in t sec.)





FIND $\frac{dx}{dt}$ WHEN $t = \frac{1}{2}$

$s = 16t^2 \Rightarrow \frac{ds}{dt} = 32t$

QUANTITIES:
x, s

SIMILAR Δ'S (1) ~ (2)

COMMON RATIO: $\frac{\text{BASE}}{\text{HEIGHT}} = \frac{30}{s} = \frac{x-30}{50-s}$

$$30(50 - s) = s(x - 30)$$

$$1500 - \cancel{30s} = sx - \cancel{30s}$$

$$sx = 1500$$

RATES:

DIFFERENTIATE

$$\frac{d}{dt}[sx] = \frac{d}{dt}[1500]$$

$$\frac{ds}{dt}x + s\frac{dx}{dt} = 0$$

PRODUCT RULE!

$$\frac{dx}{dt} = \frac{-x\frac{ds}{dt}}{s}$$

FIND $\left.\frac{dx}{dt}\right|_{t=\frac{1}{2}}$

WHEN $t = \frac{1}{2}$:

$$s(t) = 16t^2$$

$$s(\tfrac{1}{2}) = 16(\tfrac{1}{2})^2 = 4$$

$$\frac{ds}{dt} = 32t \longrightarrow \left.\frac{ds}{dt}\right|_{t=\frac{1}{2}} = s'(\tfrac{1}{2}) = 16$$

$$30(50-s) = s(x-30)$$

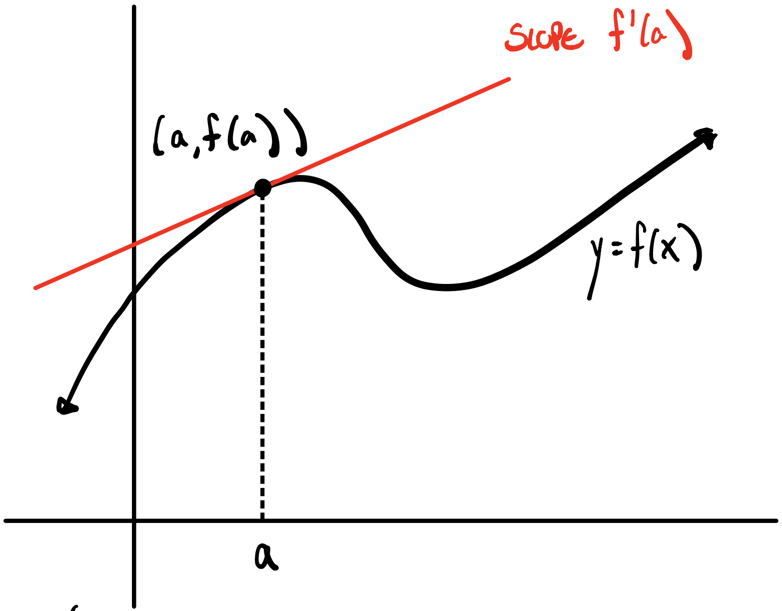
$$30(50-4) = 4(x-30)$$

$$1500 - 1\cancel{20} = 4x - 1\cancel{20}$$

$$x = 375$$

$$\frac{dx}{dt} = \frac{-x\frac{ds}{dt}}{s} = \frac{-(375)(16)}{4} = \boxed{-1500 \text{ ft/s}}$$

§2.9 LINEAR APPROXIMATION & DIFFERENTIALS



(Line thru (a,b) with slope m: $y - b = m(x-a)$)

- **TANGENT LINE** to $y = f(x)$ at $(a, f(a))$:

$$y - f(a) = f'(a)(x-a)$$

$$y = \underbrace{f(a) + f'(a)(x-a)}_{L(x)}$$

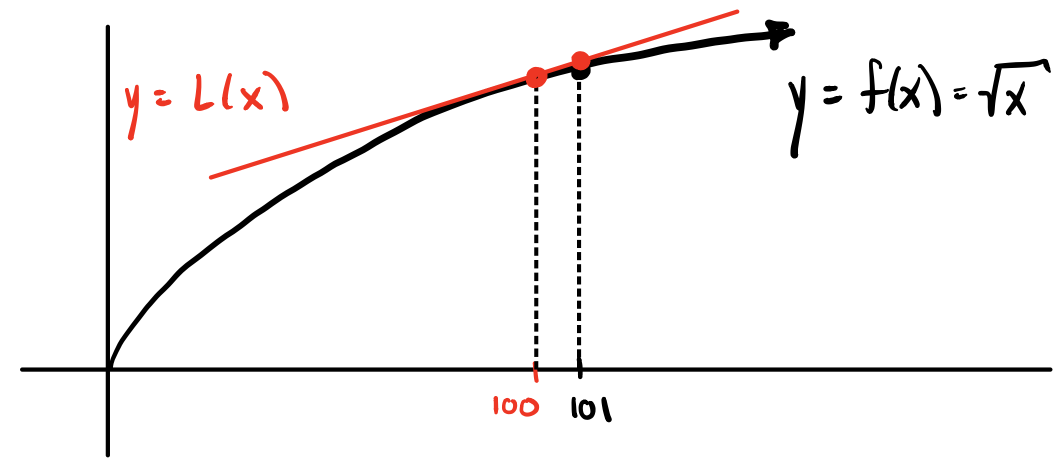
$L(x)$ IS THE <u>LINEAR</u> FUNCTION THAT DOES THE BEST JOB OF APPROXIMATING THE DIFFERENTIABLE FUNCTION $f(x)$ WHEN INPUT x IS NEAR a.

$L(x) = f(a) + f'(a)(x-a)$ IS CALLED THE **LINEARIZATION** OF f AT a.

$\underbrace{f(x) \approx L(x)}$ WHEN $|x-a|$ IS SMALL.

LINEAR APPROXIMATION

ex. USE LINEAR APPROXIMATION TO ESTIMATE $\sqrt{101}$.



$$L(x) = \underbrace{f(100)}_{\sqrt{100}=10} + f'(100)(x-100)$$

LINEARIZATION OF $f(x)$ AT 100.

$$f(x) = x^{1/2}$$

$$f'(x) = \frac{1}{2}x^{-1/2}$$

$$f'(100) = \frac{1}{2}(100)^{-1/2} = \frac{1}{2}\left(\frac{1}{10}\right) = \frac{1}{20}$$

$$L(x) = 10 + \frac{1}{20}(x-100)$$

LINEAR APPROX: $f(101) \approx L(101)$

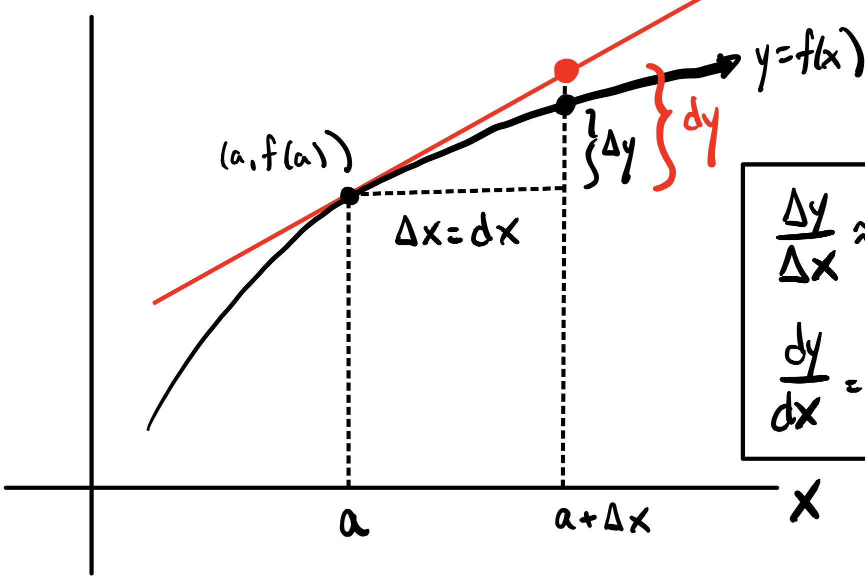
$$f(101) \approx 10 + \frac{1}{20}(101-100)$$

$$\approx \boxed{10.05}$$

$$\sqrt{101} = 10.499\ldots$$

DIFFERENTIALS

$y = L(x) = f(a) + f'(a)(x-a)$



$$\frac{\Delta y}{\Delta x} \approx f'(a) \quad : \quad \Delta y \approx f'(a)\Delta x$$

$$\frac{dy}{dx} = f'(a) \quad : \quad dy = f'(a)\, dx$$

CHANGE IN INPUT : $\Delta x = dx$ (SAME THING)

CHANGE IN OUTPUT : $\Delta y = f(a+\Delta x) - f(a)$

$$dy = L(a+\Delta x) - L(a)$$

$$dy = f'(a)\, dx$$

DEPENDENT DIFFERENTIAL VARIABLE

INDEPENDENT DIFFERENTIAL VARIABLE

$dy = L(x) - L(a)$

CHANGE IN OUTPUT OF THE LINEARIZATION OF f.

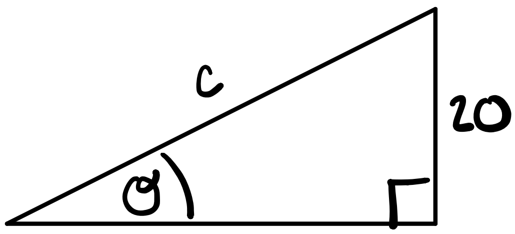
$dx = x - a$

CHANGE IN INPUT

One side of a right triangle is known to be 20 cm long and the opposite angle is measured as 30°, with a possible error of ±1°.

(a) Use differentials to estimate the error in computing the length of the hypotenuse.

(b) What is the percentage error?



$$\sin\theta = \frac{20}{c}$$

$$c = \frac{20}{\sin\theta} = 20\csc\theta$$

$$\frac{dc}{d\theta} = -20\csc\theta\cot\theta$$

NOW W/ DIFFERENTIALS: $dc = -20\csc\theta\cot\theta\, d\theta$

WHEN $\theta = 30^\circ = \frac{\pi}{6}$ RADIANS

& $d\theta = \pm 1^\circ = \pm\frac{\pi}{180}$ RADIANS

$$dc = -20\csc\frac{\pi}{6}\cot\frac{\pi}{6}\left(\pm\frac{\pi}{180}\right)$$

$$= \pm(20)(2)(\sqrt{3})\left(\frac{\pi}{180}\right) = \pm 1.2092 \text{ cm}$$

RELATIVE ERROR / PERCENT ERROR:

$$\text{REL. ERROR} = \frac{\text{ERROR}}{\text{VALUE}} = \frac{\pm 1.2092}{40} = .0302\ldots$$

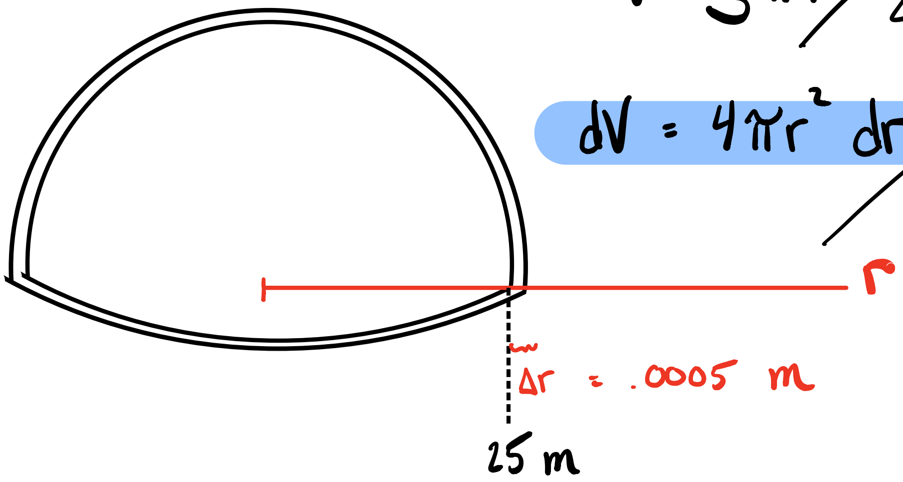
PERCENT ERROR = 3.02%

$c = 20\csc\theta = 20\csc\left(\frac{\pi}{6}\right) = 20(2) = 40$

Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with diameter 50 m.

$V = \dfrac{\frac{4}{3}\pi r^3}{2}$

$\boxed{dV = \dfrac{4\pi r^2\,dr}{2}}$ $\quad \left(\dfrac{dV}{dr} = \dfrac{4\pi r^2}{2}\right)$



set $r = 25$, $dr = .0005$

$$dV = \frac{4\pi(25)^2(.0005)}{2} = \frac{3.927\ m^3}{2}$$

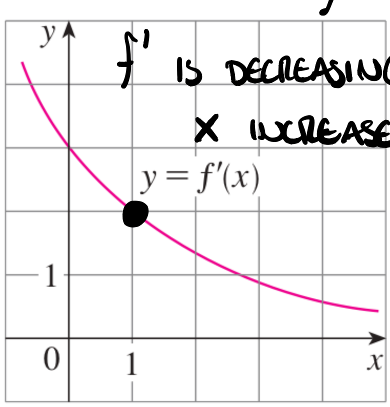
Suppose that the only information we have about a function f is that $f(1) = 5$ and the graph of its *derivative* is as shown.

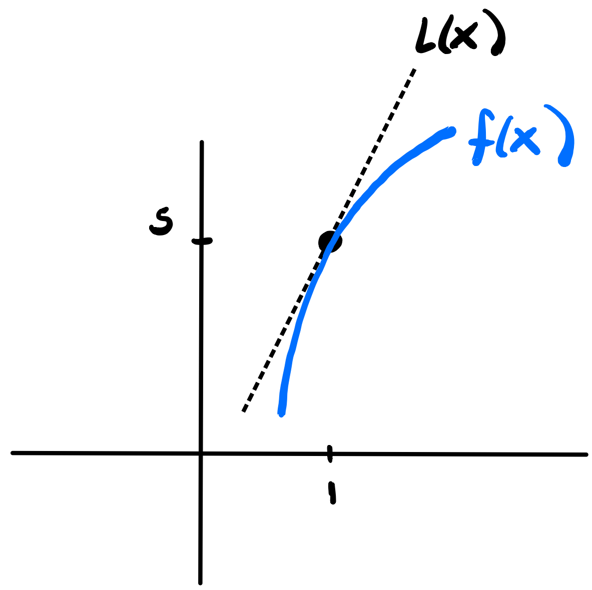
(a) Use a linear approximation to estimate $f(0.9)$ and $f(1.1)$.

(b) Are your estimates in part (a) too large or too small? Explain.

SLOPE OF f IS DECREASING

f' IS DECREASING AS X INCREASES





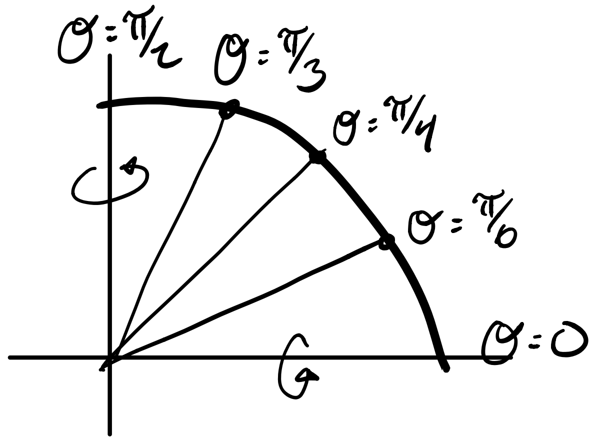
$$f(x) \approx L(x) = f(a) + f'(a)(x-a) \quad , \quad a = 1$$
$$\approx f(1) + f'(1)(x-1)$$
$$\approx 5 + 2(x-1)$$

$$f(.9) \approx 5 + 2(.9-1) = 5 - .2 = 4.8$$
$$f(1.1) \approx 5 + 2(1.1-1) = 5 + .2 = 5.2$$

LINEARIZATION $L(x) > f(x)$

$\Rightarrow$ APPROXIMATIONS WERE OVERESTIMATES.

θ	y $\sin\theta$	x $\cos\theta$
0	0	1
$\pi/6$	$1/2$	$\sqrt{3}/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$	$1/2$
$\pi/2$	1	0



$$(x^2+y^2)^2 = \frac{25}{2}(x^2-y^2)$$

Find the points on the lemniscate where the tangent is horizontal.

$$2\cancel{8}(x^2+y^2)^2 = \overset{25}{\cancel{81}}(x^2-y^2) \qquad x^2+y^2 = \pm\sqrt{\frac{25}{2}(x^2-y^2)}$$

$$2\frac{d}{dx}\left[(x^2+y^2)^2\right] = 25\frac{d}{dx}\left[x^2-y^2\right]$$

$$2\cdot 2(x^2+y^2)^{2-1}\frac{d}{dx}\left[x^2+y^2\right] = 25\left(2x-2y\frac{dy}{dx}\right)$$

$$4(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right) = 25\left(2x-2y\frac{dy}{dx}\right)$$

$$8x(x^2+y^2) + 8y(x^2+y^2)\frac{dy}{dx} = 50x-50y\frac{dy}{dx}$$

$$8y(x^2+y^2)\frac{dy}{dx} + 50y\frac{dy}{dx} = 50x - 8x(x^2+y^2)$$

$$\frac{dy}{dx} = \frac{50x-8x(x^2+y^2)}{8y(x^2+y^2)+50y} = 0$$

$$50x-8x(x^2+y^2) = 0$$

$$2x\left[25-4(x^2+y^2)\right] = 0$$

$$x=0 \qquad x^2+y^2 = \frac{25}{4} \rightarrow y = \pm\sqrt{\frac{25}{4}-x^2}$$

$$2(x^2+y^2)^2 = 25(x^2-y^2)$$

$$2\left(\frac{25}{4}\right)^2 = 25(x^2-y^2)$$

$$\frac{25^2}{8} = 25(x^2-y^2)$$

$$\frac{25}{8} = x^2-y^2$$

$$x^2+y^2 = \frac{25}{4}$$

$$+\quad x^2-y^2 = \frac{25}{8}$$

$$2(x^2+y^2)^2 = 25(x^2-y^2)$$

$$2\left(\frac{75}{16}+y^2\right)^2 = 25\left(\frac{75}{16}-y^2\right)$$

$$2x^2 = \frac{75}{8}$$

$$x^2 = \frac{75}{16} \rightarrow x = \pm\frac{5\sqrt{3}}{4}$$

$$x^2 = \frac{75}{16}$$

$$y^2 = \frac{25}{4} - x^2 = \frac{25}{4} - \frac{75}{16} = \frac{25}{16}$$

$y = \pm \frac{5}{4}$

$\left(\pm \frac{5\sqrt{3}}{4} , \pm \frac{5}{4} \right)$