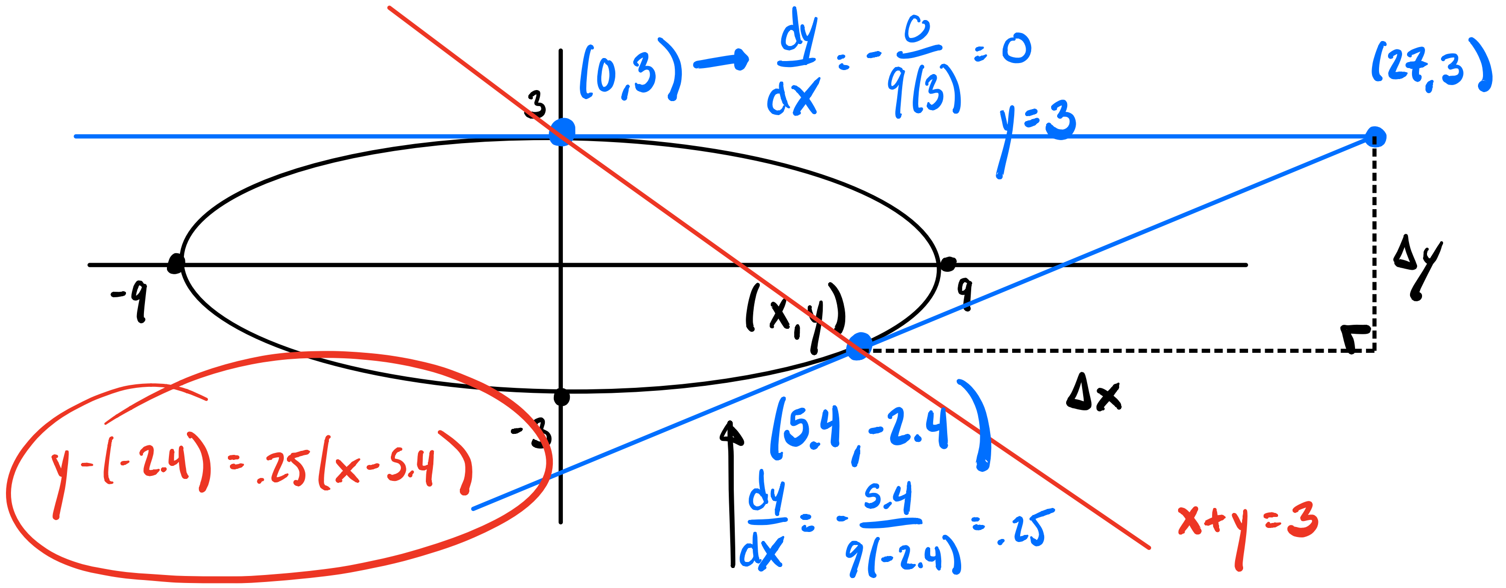
Find equations of both the tangent lines to the ellipse $x^2 + 9y^2 = 81$ that pass through the point $(27, 3)$.

$y =$ $\boxed{3}$ (smaller slope)

$y =$ $\boxed{\frac{1}{4}x - \frac{15}{4}}$ (larger slope)

ELLIPSE: $\frac{x^2}{9^2} + \frac{y^2}{3^2} = 1$



SLOPE OF TANGENT LINE: $\frac{\Delta y}{\Delta x} = \frac{3-y}{27-x}$

$$\frac{d}{dx}\left[x^2 + 9y^2 = 81\right]$$

$$2x + 18y\frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = \frac{-2x}{18y} = -\frac{x}{9y}$$

WHAT DO WE KNOW ABOUT (x,y)?

(1) $x^2 + 9y^2 = 81$ (ON ELLIPSE)

(2) $\frac{3-y}{27-x} = -\frac{x}{9y} \rightarrow 9y(3-y) = -x(27-x)$

$$27y - 9y^2 = x^2 - 27x$$

$$27x + 27y = x^2 + 9y^2 = 81$$

$$27(x+y) = 81$$

$$x + y = 3$$

$$y = 3 - x$$

$$\frac{3-(3-x)}{27-x} = -\frac{x}{9(3-x)}$$

$$\frac{x}{27-x} = \frac{-x}{27-9x} \Rightarrow x(27-9x) = -x(27-x)$$

$$27x - 9x^2 = -27x + x^2$$

$$54x = 10x^2$$

$$0 = 10x^2 - 54x$$

$$0 = x(10x - 54)$$

$x = 0$ $\quad\quad$ $x = 5.4$

$y = 3 - (0) = 3$ $\quad\quad$ $y = 3 - (5.4) = -2.4$

$(0, 3)$ $\quad\quad$ $(5.4, -2.4)$

§3.1 Maximum & Minimum Values

- Definitions: Absolute vs. Local
- Extreme Value Theorem
- Fermat's Theorem (Do's & Don'ts)
- Critical Numbers
- The Closed Interval Method

Def: A point $c \in \mathrm{Dom}(f)$ is an absolute maximum of f if

$$f(c) \geq f(x) \quad \forall x \in \mathrm{Dom}(f),$$

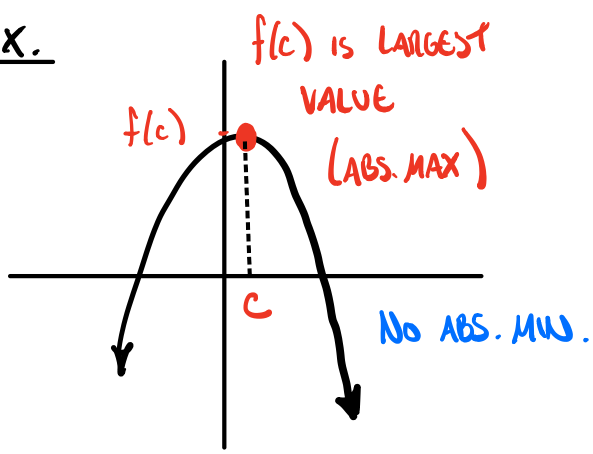
and we call $f(c)$ the absolute maximum value.

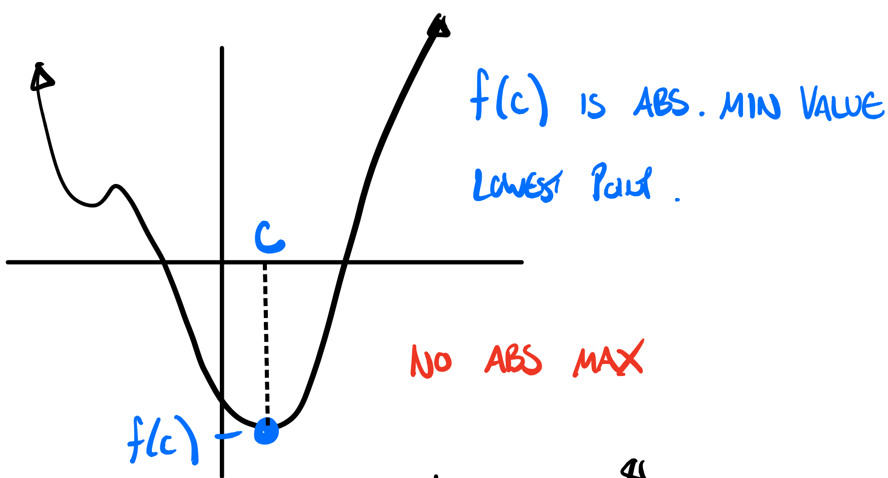
A point $c \in \mathrm{Dom}(f)$ is an absolute minimum of f if

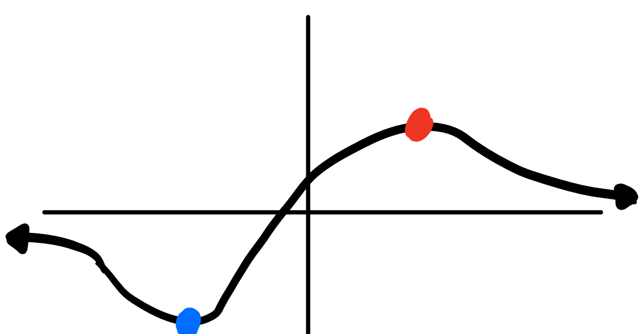
$$f(c) \leq f(x) \quad \forall x \in \mathrm{Dom}(f),$$

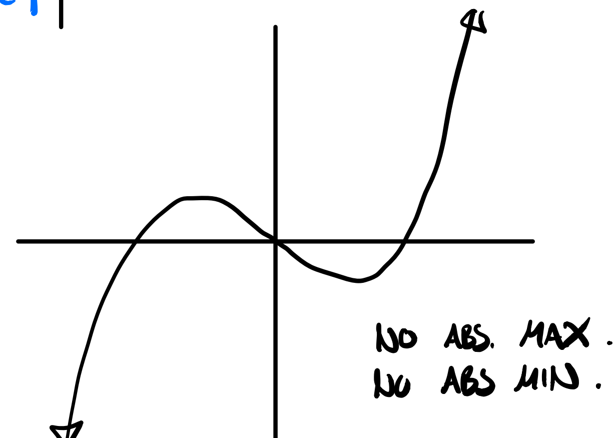
and we call $f(c)$ the absolute minimum value.

ex.

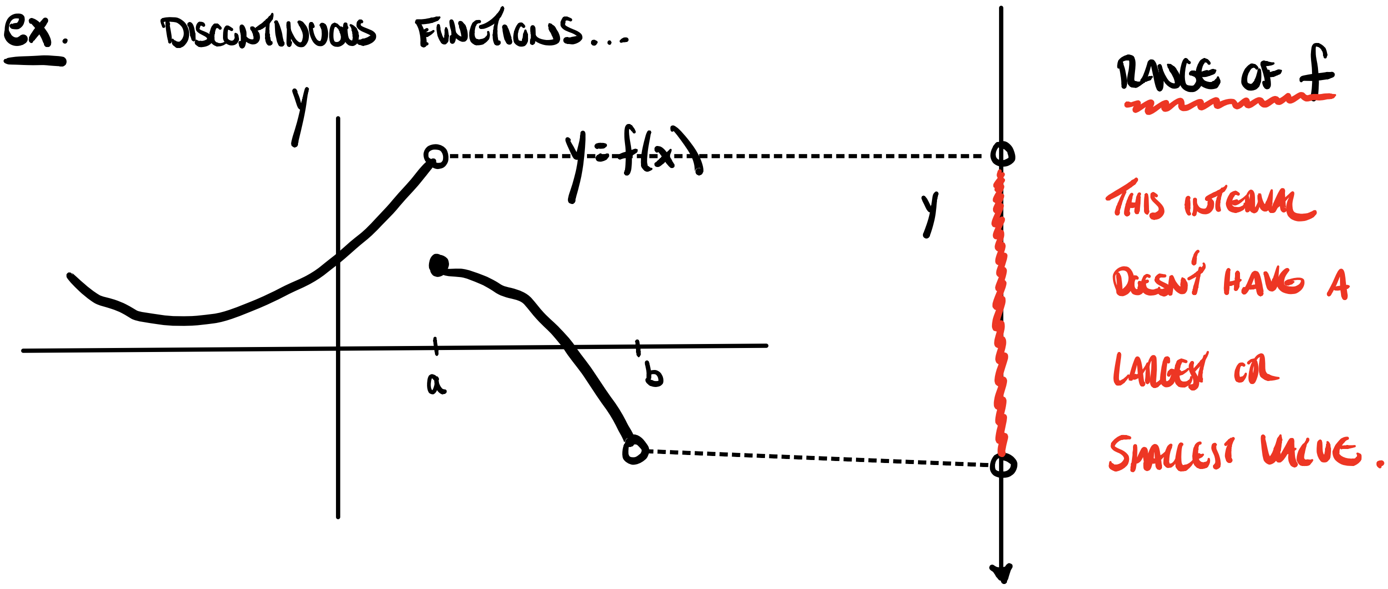




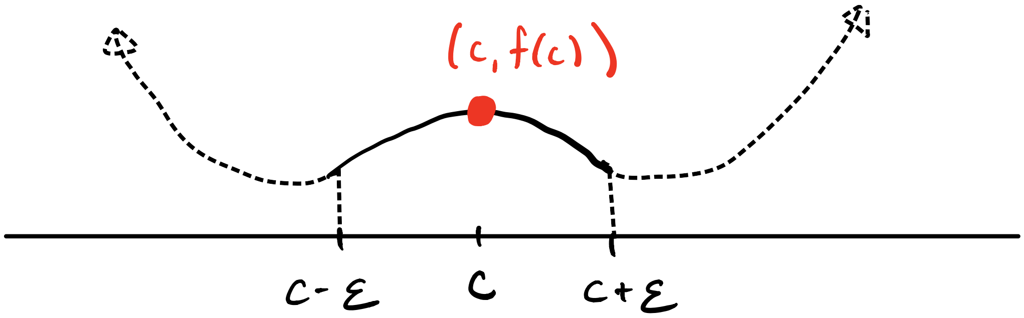




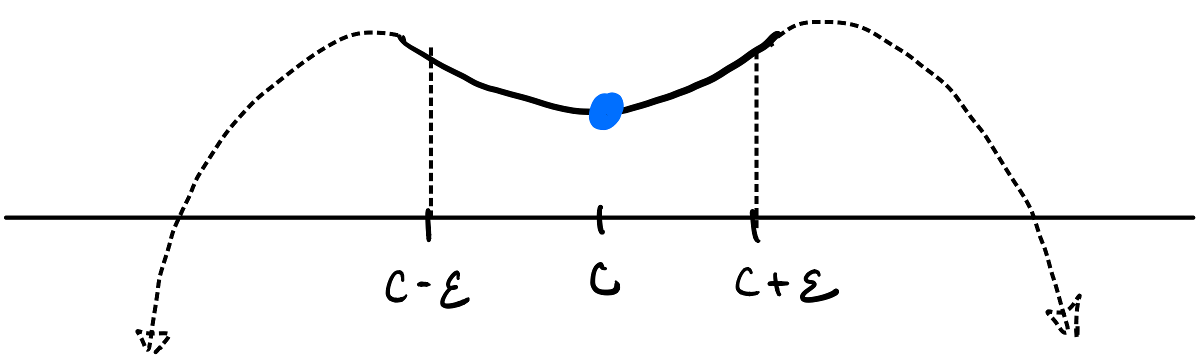
ex. DISCONTINUOUS FUNCTIONS...



Def: A POINT $c \in Dom(f)$ IS A LOCAL MAXIMUM OF f IF $\exists\, \varepsilon > 0$ s.t. $f(c) \geq f(x) \quad \forall x \in (c-\varepsilon, c+\varepsilon)$, AND WE CALL $f(c)$ A LOCAL MAXIMUM VALUE.

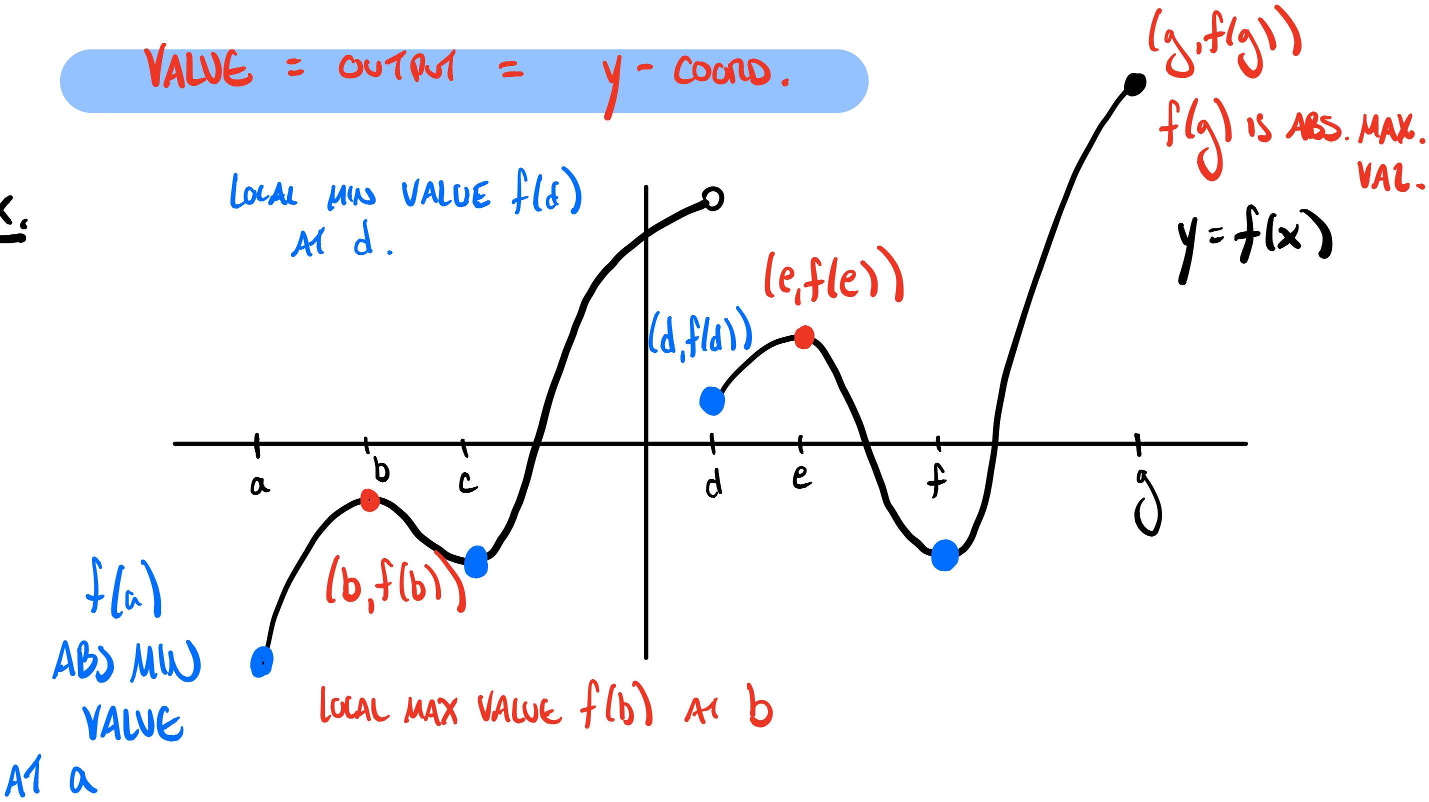


A POINT $c \in Dom(f)$ IS A LOCAL MINIMUM OF f IF $\exists\, \varepsilon > 0$ s.t. $f(c) \leq f(x) \quad \forall x \in (c-\varepsilon, c+\varepsilon)$, AND WE CALL $f(c)$ A LOCAL MINIMUM VALUE.



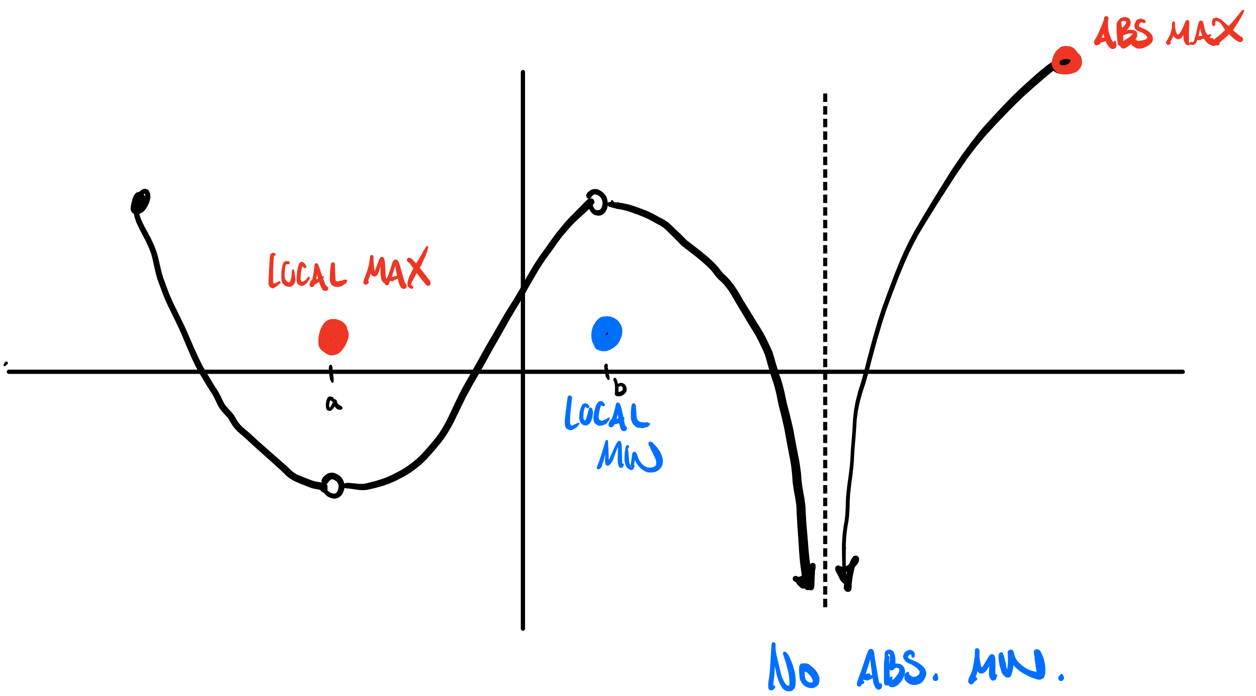
VALUE = OUTPUT = y - COORD.

ex.



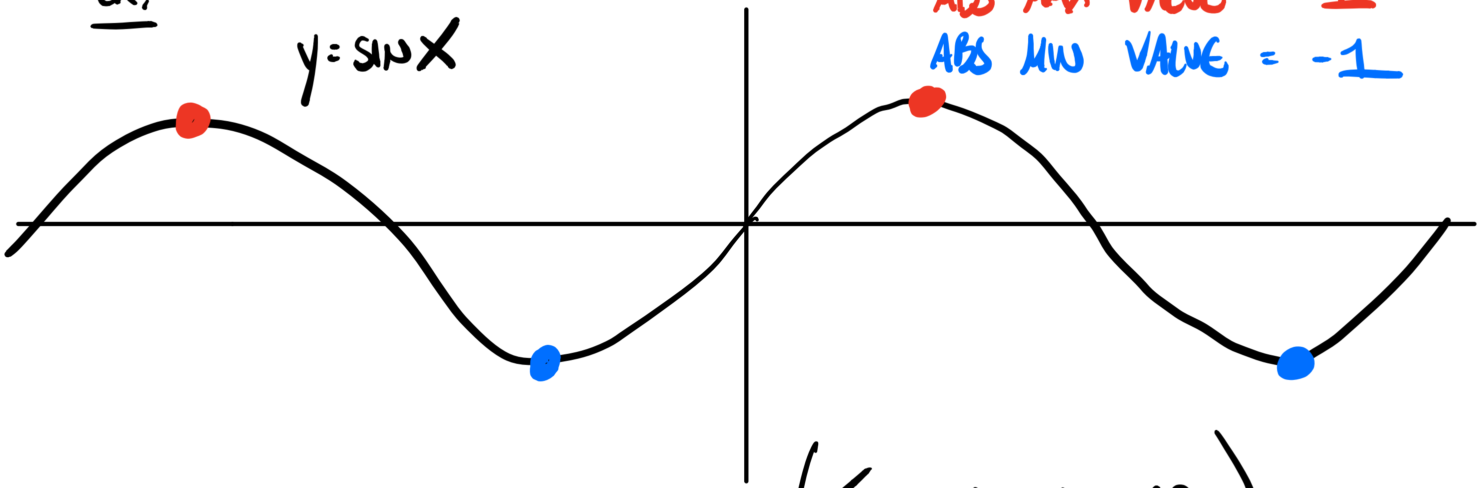
NOTE: LOCAL MAX/MIN VALUES OF f CANNOT OCCUR AT ENDPOINTS OF THE DOMAIN OF f.

ex.



NOTE: THERE CAN BE MANY LOCAL EXTREME VALUES, BUT ONLY ONE ABS MAX & ONE ABS MIN (AT MOST) ex.

ABS MAX VALUE = 1
ABS MIN VALUE = -1

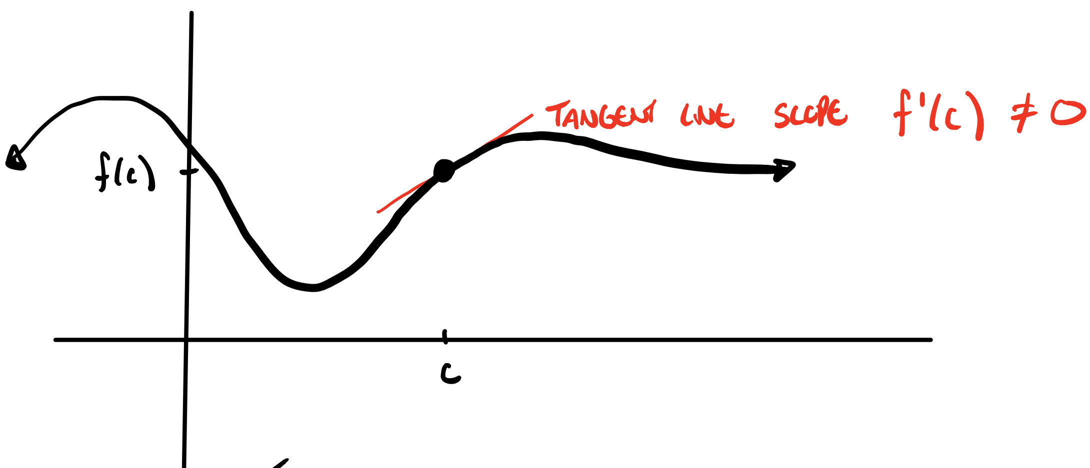


(TIES ARE ALLOWED.)

4 Fermat's Theorem If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

WHY? SUPPOSE $f'(c) \neq 0$.

↳ TANGENT LINE TO $y = f(x)$ AT $(c, f(c))$ HAS NON-ZERO SLOPE.



THEN BY PLUGGING IN VALUES $x > c$ & $x < c$ WE GET OUTPUTS $f(x)$ BOTH $f(x) > f(c)$ & $f(x) < f(c)$.

$\Rightarrow$ $f(c)$ IS NOT LOCAL MAX OR LOCAL MIN.

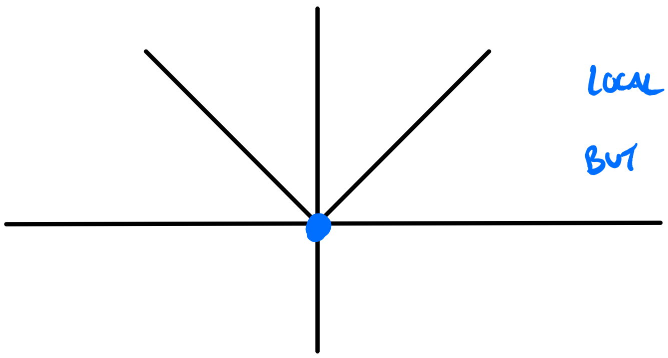
IF $f'(c)$ EXISTS & $f'(c) \neq 0$ THEN $f(c)$ IS NOT A LOCAL EXTREME VALUE.

(SAME THING AS FERMAT'S THM). ✓

- C IS A LOCAL EXTREME VALUE & $f'(c)$ EXISTS $\Rightarrow f'(c) = 0$

- C IS A LOCAL EXTREME VALUE $\not\Rightarrow f'(c) = 0$

  e.g. $f(x) = |x|$



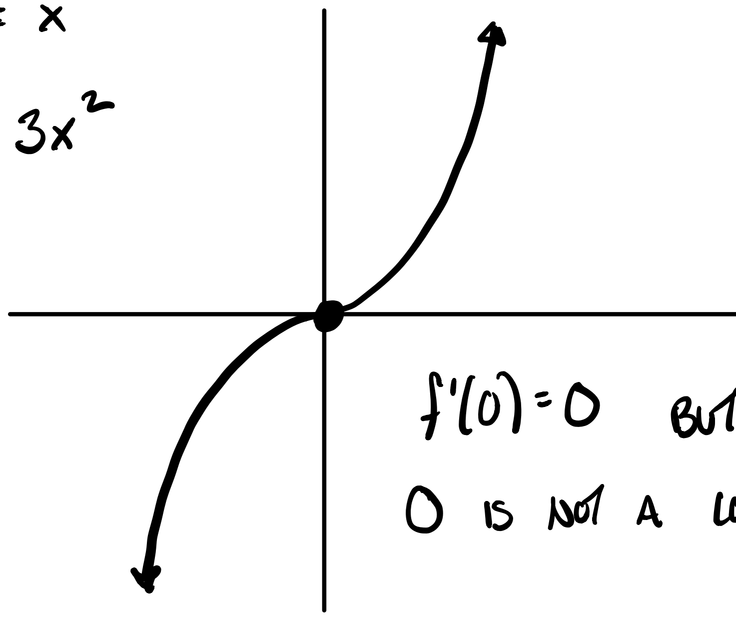
LOCAL MIN VALUE 0 AT 0

BUT $f'(0)$ D.N.E.

- $f'(c) = 0 \not\Rightarrow$ C IS A LOCAL EXTREME VALUE

  e.g. $f(x) = x^3$

  $f'(x) = 3x^2$



$f'(0) = 0$ BUT

0 IS NOT A LOCAL MAX/MIN.

ALTERNATE FERMAT'S THM:

IF $f(c)$ IS A LOCAL MAX/MIN VALUE

THEN EITHER $f'(c) = 0$ OR

$f'(c)$ D.N.E.

6 Definition A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

AKA "CRITICAL POINT"

ALTERNATE FERMAT'S THM:

IF $f(c)$ IS A LOCAL MAX/MIN VALUE

THEN C IS A CRITICAL POINT OF f.

29–42 Find the critical numbers of the function.

29. $f(x) = 4 + \frac{1}{3}x - \frac{1}{2}x^2$

30. $f(x) = x^3 + 6x^2 - 15x$

31. $f(x) = 2x^3 - 3x^2 - 36x$

32. $f(x) = 2x^3 + x^2 + 2x$

33. $g(t) = t^4 + t^3 + t^2 + 1$

34. $g(t) = |3t - 4|$

* **39.** $F(x) = x^{4/5}(x - 4)^2$

41. $f(\theta) = 2\cos\theta + \sin^2\theta$

WE SEEK #'S X SUCH THAT EITHER $F'(x) = 0$ OR $F'(x)$ D.N.E.

$$F'(x) = \frac{4}{5}x^{-1/5}(x-4)^2 + x^{4/5}\cdot 2(x-4)$$

$$F'(x) = 2x^{-1/5}(x-4)\left[\frac{2}{5}(x-4) + x\right]$$

$$= \frac{2(x-4)\left[\frac{2}{5}(x-4) + x\right]}{x^{1/5}}$$

$F'(x) = 0$ WHEN $x - 4 = 0 \rightarrow x = 4$, OR

$$\frac{2}{5}x - \frac{8}{5} + x = 0$$

$$\frac{7}{5}x = \frac{8}{5} \rightarrow x = \frac{8}{7}$$

$F'(4) = 0$

$F'\left(\frac{8}{7}\right) = 0$

$F'(0)$ D.N.E. (0 IS IN DOM(F))

$$\boxed{\text{CRITICAL \#'S } 0, \tfrac{8}{7}, 4}$$