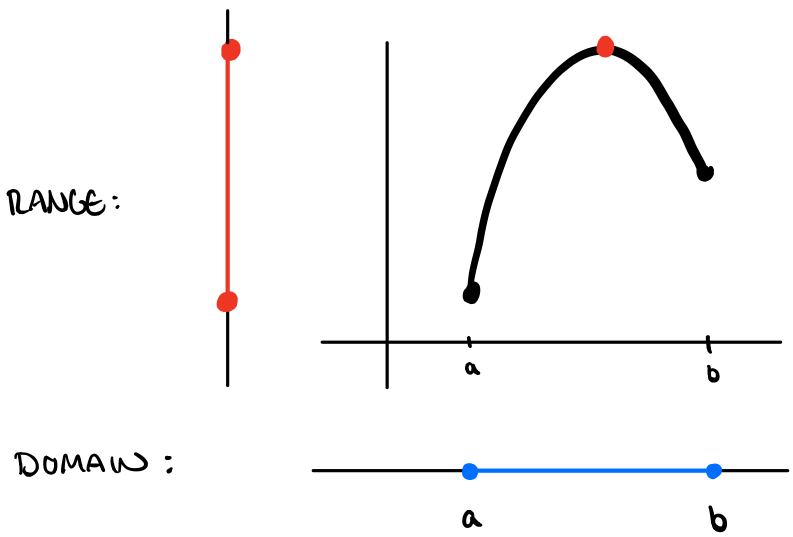
# §3.1 Maximum & Minimum Values

One missing piece from this section:

**3 The Extreme Value Theorem** If $f$ is continuous on a closed interval $[a, b]$, then $f$ attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers $c$ and $d$ in $[a, b]$.

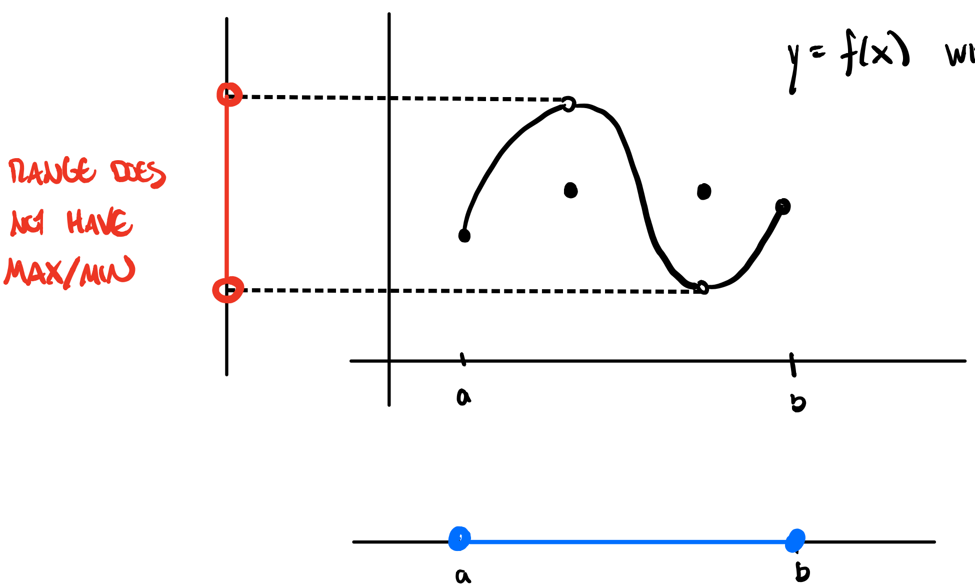
Very intuitive, but the conditions are necessary.

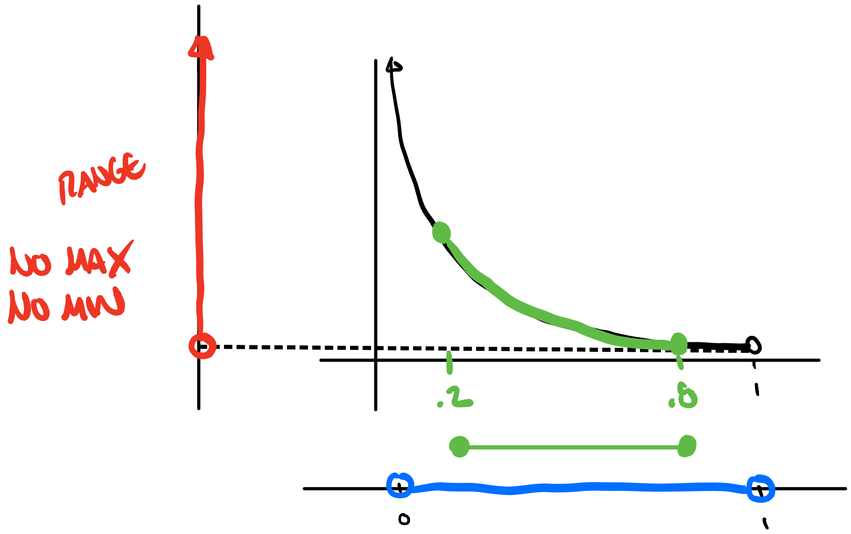



Range is closed interval.

It has max/min value

y = f(x) with discontinuities.






$y = \frac{1}{x}$ ON $(0,1)$

$f$ WITH OPEN DOMAIN

PROBLEM GOES AWAY ON CLOSED DOMAIN

# §3.2 The Mean Value Thm.

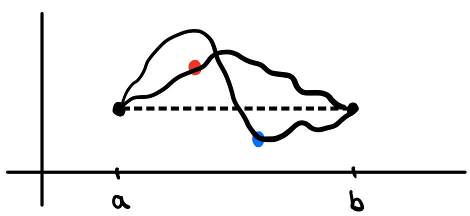
**Rolle's Theorem** Let $f$ be a function that satisfies the following three hypotheses:

1. $f$ is continuous on the closed interval $[a, b]$. ← EXTREME VALUE THM CAN BE APPLIED!
2. $f$ is differentiable on the open interval $(a, b)$.
3. $f(a) = f(b)$

Then there is a number $c$ in $(a, b)$ such that $f'(c) = 0$.

PROOF: 1. IF $f(x) = k$ CONSTANT $\Rightarrow$ OBVIOUS.

2. ELSE, $f(x) \gtrless f(a) = f(b)$ FOR SOME $x \in (a,b)$. $(*)$



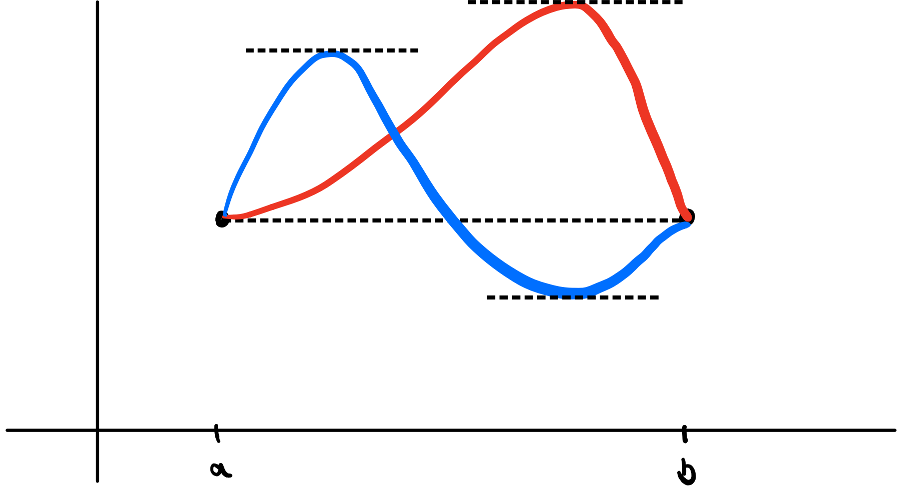

EXTREME VALUE THM $\Rightarrow \exists\, c \in [a,b]$ SUCH THAT

$f(c)$ IS AN ABSOLUTE MAX/MIN.

$(*) \Rightarrow c \in (a,b) \Rightarrow c$ IS LOCAL MAX/MIN.

FERMAT'S THM $\Rightarrow f'(c) = 0$.

$\square$




QUESTION: WHEN FUNCTIONS ARE DEFINED ON CLOSED INTERVALS $[a,b]$ AND DIFFERENTIABLE ON OPEN INTERVALS $(a,b)$, WHY CAN'T THEY BE DIFFERENTIABLE AT ENDPOINTS $a$ & $b$?

$\underbrace{\text{DERIVATIVE OF } f \text{ AT } a}$, $\quad \mathrm{DOM}(f) = [a,b]$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{ONLY EXISTS IF}$$

$$\lim_{\substack{x \to a^- \\ x < a}} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}$$

$f$ IS NOT DEFINED HERE.

**The Mean Value Theorem** Let $f$ be a function that satisfies the following hypotheses:

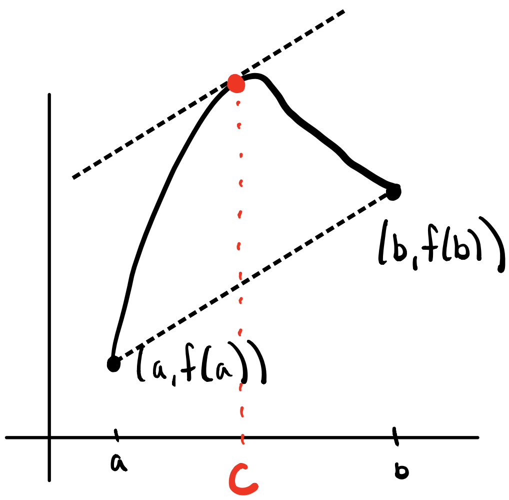
1. $f$ is continuous on the closed interval $[a, b]$.
2. $f$ is differentiable on the open interval $(a, b)$.

Then there is a number $c$ in $(a, b)$ such that

$$\boxed{1} \qquad f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$\boxed{2} \qquad f(b) - f(a) = f'(c)(b - a)$$




SLOPE OF SECANT LINE FROM $(a, f(a))$ TO $(b, f(b))$

$$\frac{f(b) - f(a)}{b - a} \leftarrow \text{AVERAGE RATE OF CHANGE}$$

EQUALS SLOPE OF TANGENT LINE SOMEWHERE BETWEEN.

$$f'(c) \leftarrow \text{INSTANTANEOUS RATE OF CHANGE}$$

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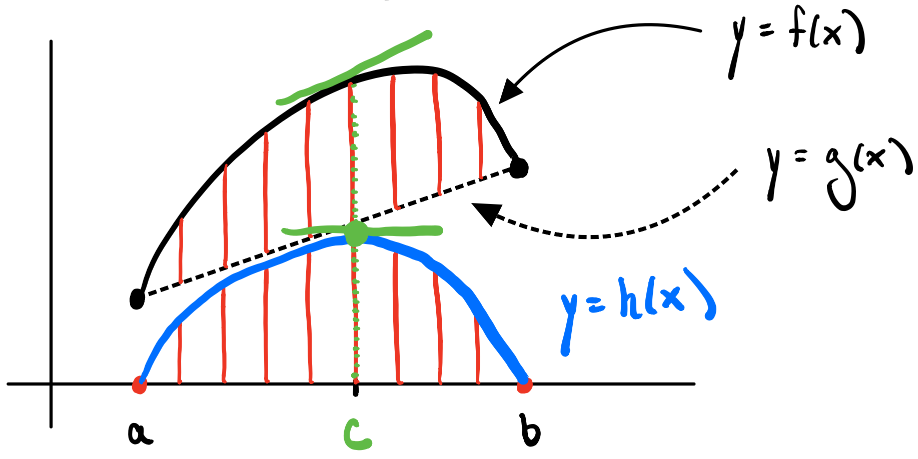
PROOF: SET
$$\underbrace{g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)}_{\text{GRAPH IS LINE THROUGH } (a, f(a)) \text{ \& } (b, f(b))}$$

SET $h(x) = f(x) - g(x)$

$h'(c) = 0$



1. $h$ is continuous on $[a,b]$
2. $h$ is differentiable on $(a,b)$
3. $h(a) = h(b) = 0$

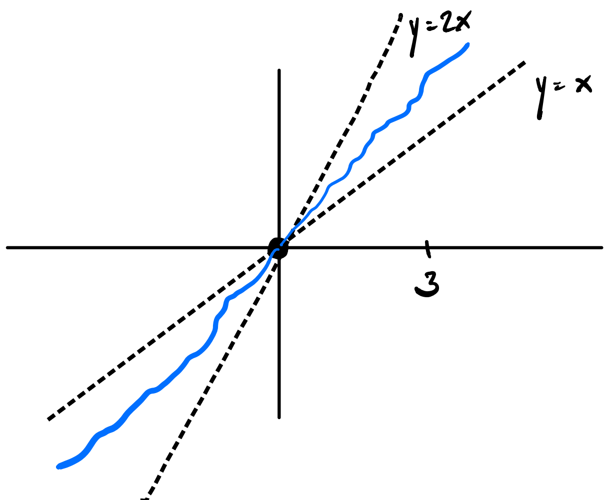
Rolles' thm $\Rightarrow$ $\exists\, c \in (a,b)$ such that $h'(c) = 0$.

Since $h'(c) = f'(c) - g'(c)$ we have

$$f'(c) = g'(c) = \frac{f(b) - f(a)}{b - a}. \qquad \square$$

ex. Suppose $f(0) = 0$ & $1 \le f'(x) \le 2$ for all $x \in \mathbb{R}$.

What are the possible values for $f(3)$? For $f(a)$?




$f$ is cont. on $[0,3]$

$f$ is diff. on $(0,3)$

MVT $\Rightarrow$ $\dfrac{f(3) - f(0)}{3 - 0} = f'(c)$ for some $c \in (0,3)$

$$f(3) = 3f'(c)$$

$$3 = 3 \cdot 1 \le f(3) = 3f'(c) \le 3 \cdot 2 = 6$$

$$3 \le f(3) \le 6$$

MORE GENERALLY, $f(a) = a f'(c)$

$$a \le f(a) \le 2a \qquad \text{IF} \qquad a > 0$$
$$2a \le f(a) \le a \qquad \text{IF} \qquad a < 0$$

**31.** Use the Mean Value Theorem to prove the inequality

$$|\sin a - \sin b| \leqslant |a - b| \qquad \text{for all } a \text{ and } b$$

EQUIVALENTLY: SHOW $\dfrac{|\sin a - \sin b|}{|a - b|} \le 1$

$$\left| \frac{\sin a - \sin b}{a - b} \right| \le 1 \qquad -1 \le \frac{\sin a - \sin b}{a - b} \le 1$$

$f(x) = \sin x$ IS 1. CONT. ON $[a,b]$ (ASSUME $a \neq b$)

2. DIFF ON $(a,b)$

MVT $\Rightarrow$ $\dfrac{f(a) - f(b)}{a - b} = f'(c)$ FOR SOME $c \in (a,b)$

$$\left( \frac{f(a) - f(b)}{a - b} = \frac{f(b) - f(a)}{b - a} = f'(c) \right)$$

$$-1 \le f'(x) = \cos(x) \le 1 \qquad \text{FOR ALL } x$$

$\Rightarrow$

$$-1 \le \frac{f(a) - f(b)}{a - b} = f'(c) \le 1$$

✓

**5 Theorem** If $f'(x) = 0$ for all $x$ in an interval $(a, b)$, then $f$ is constant on $(a, b)$.

PROOF: $\forall\, x_1, x_2$ SATISFYING $a < x_1 < x_2 < b$

1. $f$ IS CONTINUOUS ON $[x_1, x_2]$
2. $f$ IS DIFFERENTIABLE ON $(x_1, x_2)$

MVT $\Rightarrow \exists\, c \in (x_1, x_2)$ SUCH THAT

$$f(x_2) - f(x_1) = \underbrace{f'(c)}_{= 0}(x_2 - x_1)$$

$\therefore\ f(x_1) = f(x_2)\ \forall\, x_1, x_2 \in (a, b)$,

i.e. $f$ IS CONSTANT ON $(a, b)$. $\square$

( NOTE: IF WE ASSUME $f$ IS CONTINUOUS ON $[a, b]$

AND $f'(x) = 0\ \forall x \in (a, b)$

THEN $f(x)$ IS CONSTANT ON $[a, b]$. )

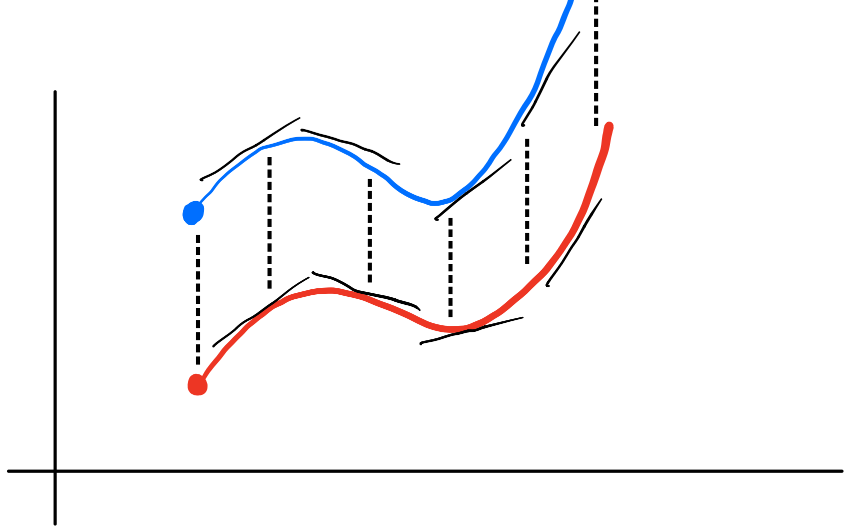
**7 Corollary** If $f'(x) = g'(x)$ for all $x$ in an interval $(a, b)$, then $f - g$ is constant on $(a, b)$; that is, $f(x) = g(x) + c$ where $c$ is a constant.

PROOF: SET $h(x) = f(x) - g(x)$

$h'(x) = f'(x) - g(x) = 0$ ON $(a, b)$ BY ASSUMPTION.

$\Rightarrow h(x) = c$ CONSTANT ON $(a, b)$

$\therefore\ f(x) - g(x) = c \Rightarrow f(x) = g(x) + c$ $\square$



# §3.3 How Derivatives Affect the Shape of a Graph

**Increasing/Decreasing Test**

(a) If $f'(x) > 0$ on an interval, then $f$ is increasing on that interval, $I$.

(b) If $f'(x) < 0$ on an interval, then $f$ is decreasing on that interval, $I$.

Proof: Take $a, b \in I$ with $a < b$.

MVT. $\Rightarrow$ $f(b) - f(a) = f'(c)\underbrace{(b-a)}_{\text{positive}}$ for some $c \in (a,b)$.

If $f'(c) > 0$ then $f(b) - f(a) > 0$, i.e. $f(b) > f(a)$.

If $f'(c) < 0$ then $f(b) - f(a) < 0$, i.e. $f(b) < f(a)$.

Since this holds for all $a < b$ in the interval $I$,

$f$ is increasing/decreasing on $I$. $\square$