39–50 Find the limit.

39. $\lim_{x\to 0} \frac{\sin 5x}{3x}$

40. $\lim_{x\to 0} \frac{\sin x}{\sin \pi x}$

41. $\lim_{t\to 0} \frac{\tan 6t}{\sin 2t}$

42. $\lim_{\theta\to 0} \frac{\cos\theta - 1}{\sin\theta}$

43. $\lim_{x\to 0} \frac{\sin 3x}{5x^3 - 4x}$

44. $\lim_{x\to 0} \frac{\sin 3x \sin 5x}{x^2}$

45. $\lim_{\theta\to 0} \frac{\sin\theta}{\theta + \tan\theta}$

46. $\lim_{x\to 0} \csc x \sin(\sin x)$

47. $\lim_{\theta\to 0} \frac{\cos\theta - 1}{2\theta^2}$

48. $\lim_{x\to 0} \frac{\sin(x^2)}{x}$

49. $\lim_{x\to \pi/4} \frac{1 - \tan x}{\sin x - \cos x}$

50. $\lim_{x\to 1} \frac{\sin(x-1)}{x^2 + x - 2}$

$\sin^2 x + \cos^2 x = 1$

$1 + \cot^2 x = \csc^2 x$

$\tan^2 x + 1 = \sec^2 x$

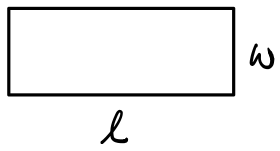
49. $$\lim_{x\to \frac{\pi}{4}} \frac{1 - \tan x}{\sin x - \cos x} = \lim_{x\to \frac{\pi}{4}} \frac{1 - \frac{\sin x}{\cos x}}{\sin x - \cos x}$$

$$= \lim_{x\to \frac{\pi}{4}} \frac{\frac{\cos x}{\cos x} - \frac{\sin x}{\cos x}}{\sin x - \cos x}$$

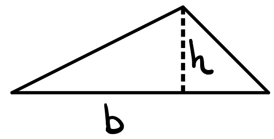
$$= \lim_{x\to \frac{\pi}{4}} \frac{1}{\sin x - \cos x}\left(\frac{\cos x - \sin x}{\cos x}\right)$$

$$= \lim_{x\to \frac{\pi}{4}} (-1)\left(\frac{1}{\cos x}\right) = \boxed{\frac{-2}{\sqrt{2}}}$$

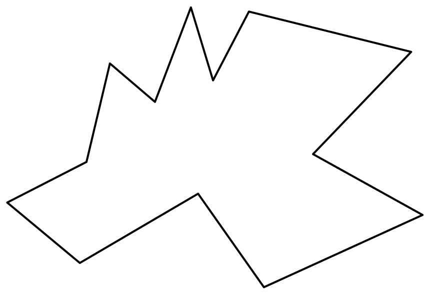
Area



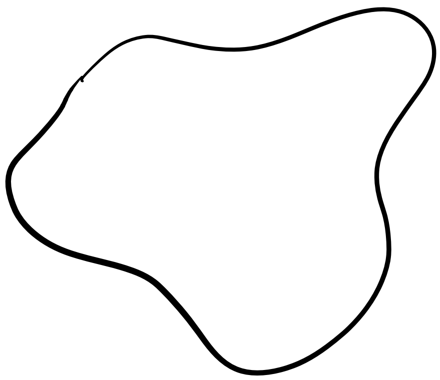
$A = \ell w$



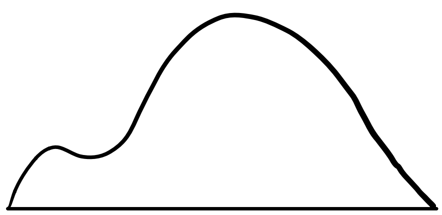
$A = \frac{1}{2} bh$



Any polygon can be triangulated.



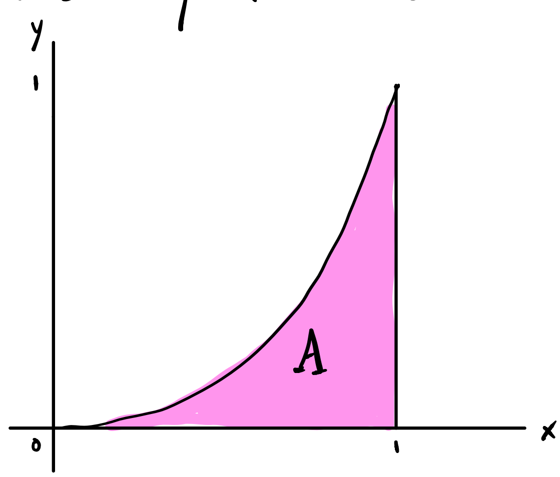
What about shapes with curved edges?

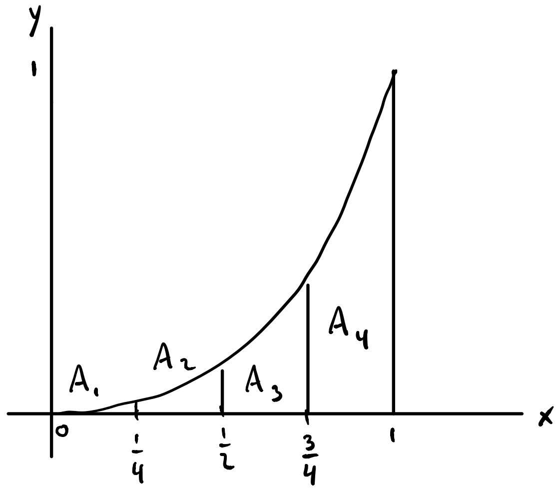


We need a way to calculate the area below a curve.

Let's make this problem precise.

Calculate the area under the curve $y = x^2$ and above the x-axis with $0 \le x \le 1$.





IF WE CUT THE INTERVAL $[0,1]$ INTO

4 SUBINTERVALS $I_1 = [0, \frac{1}{4}]$

$I_2 = [\frac{1}{4}, \frac{1}{2}]$

$I_3 = [\frac{1}{2}, \frac{3}{4}]$

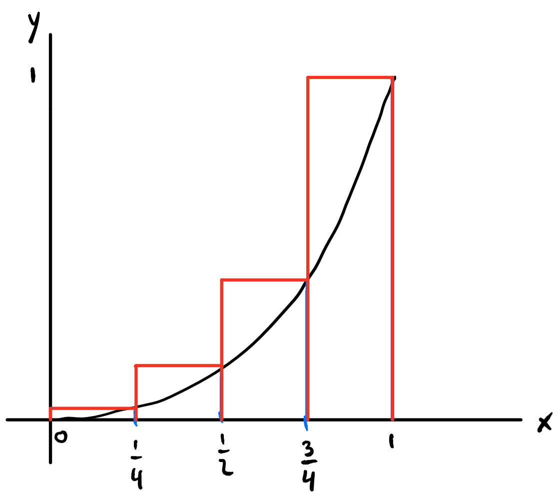
$I_4 = [\frac{3}{4}, 1]$

THE TOTAL AREA IS THE SUM OF THE AREAS UNDER THE CURVE ABOVE EACH SUBINTERVAL.

$$A = A_1 + A_2 + A_3 + A_4$$

WE CAN APPROXIMATE THE AREA A_i OF EACH VERTICAL STRIP AS THE AREA OF A RECTANGLE WITH BASE $\frac{1}{4}$

AND HEIGHT EQUAL TO THE HEIGHT OF THE CURVE AT THE RIGHT ENDPOINT OF EACH SUBINTERVAL



$$A_1 \approx \left(\tfrac{1}{4}\right)^2\left(\tfrac{1}{4}\right) = \tfrac{1}{64}$$

$$A_2 \approx \left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{4}\right) = \tfrac{4}{64}$$

$$A_3 \approx \left(\tfrac{3}{4}\right)^2\left(\tfrac{1}{4}\right) = \tfrac{9}{64}$$

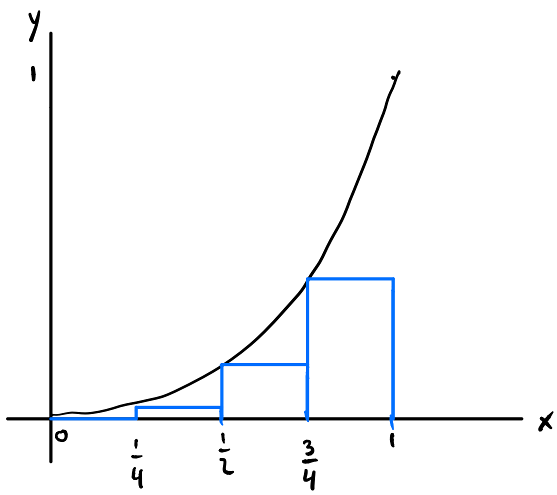
$$A_4 \approx (1)^2\left(\tfrac{1}{4}\right) = \tfrac{16}{64}$$

$$\Rightarrow A = A_1 + A_2 + A_3 + A_4$$

$$A \approx \frac{1}{64} + \frac{4}{64} + \frac{9}{64} + \frac{16}{64} = \frac{30}{64}$$

$A \approx 0.46875$ (OVERESTIMATE) ← CALL THIS R_4

ALTERNATIVELY, WE COULD HAVE APPROXIMATED THE AREA OF EACH VERTICAL STRIP AS A RECTANGLE WITH BASE $\frac{1}{4}$ AND HEIGHT EQUAL TO THE HEIGHT OF THE CURVE AT THE LEFT ENDPOINT OF EACH SUBINTERVAL



$$A_1 \approx (0)^2\left(\tfrac{1}{4}\right) = 0$$
$$A_2 \approx \left(\tfrac{1}{4}\right)^2\left(\tfrac{1}{4}\right) = 1/64$$
$$A_3 \approx \left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{4}\right) = 4/64$$
$$A_4 \approx \left(\tfrac{3}{4}\right)^2\left(\tfrac{1}{4}\right) = 9/64$$

$$\Rightarrow A = A_1 + A_2 + A_3 + A_4$$

$$A \approx 0 + \frac{1}{64} + \frac{4}{64} + \frac{9}{64} = \frac{14}{64}$$

$A \approx 0.21875$ (UNDERESTIMATE) ← CALL THIS L_4

OUR APPROXIMATION BECOMES BETTER WHEN WE MAKE THE NUMBER OF SUBINTERVALS, i.e. THE NUMBER OF RECTANGLES, n LARGER.

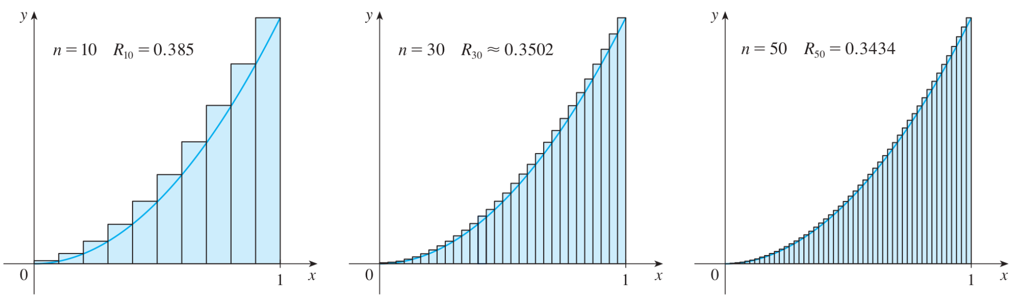


FIGURE 8 Right endpoints produce upper sums because $f(x) = x^2$ is increasing.

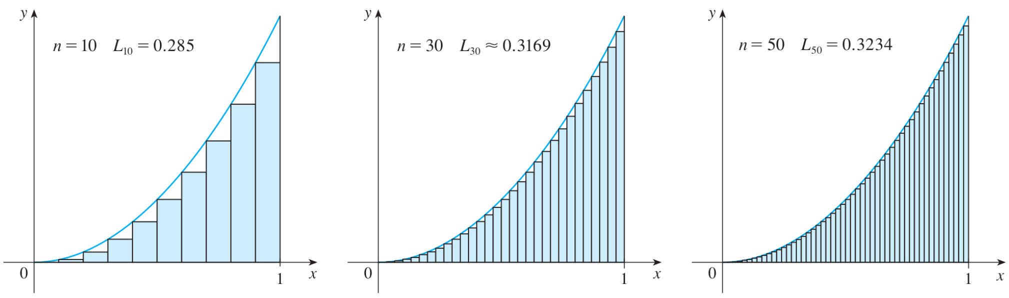


FIGURE 9 Left endpoints produce lower sums because $f(x) = x^2$ is increasing.

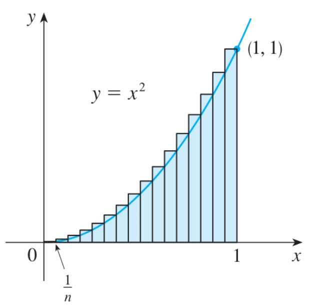
n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

INCREASING UNDERESTIMATES

DECREASING OVERESTIMATES

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n = \text{AREA}!$$

$$= \frac{1}{3}!$$



PROOF THAT $\lim_{n\to\infty} R_n = \frac{1}{3}$

$$R_n = \left(\frac{1}{n}\right)^2\left(\frac{1}{n}\right) + \left(\frac{2}{n}\right)^2\left(\frac{1}{n}\right) + \left(\frac{3}{n}\right)^2\left(\frac{1}{n}\right) + \dots + \left(\frac{n}{n}\right)^2\left(\frac{1}{n}\right)$$

$$R_n = \frac{1}{n^3}\left(1^2 + 2^2 + 3^2 + \dots + n^2\right)$$

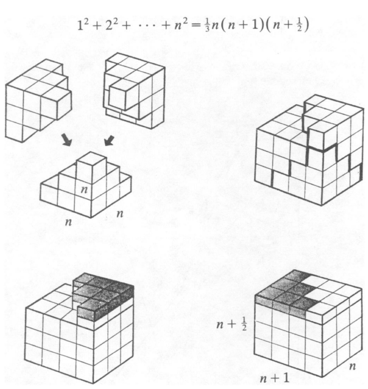
$$R_n = \frac{1}{n^3} \cdot \frac{2n^3 + 3n^2 + n}{6}$$

$$R_n = \frac{2n^3}{6n^3} + \frac{3n^2}{6n^3} + \frac{n}{6n^3}$$

$$R_n = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} = \frac{1}{3} \qquad \square$$

Proof without words:
Sum of squares

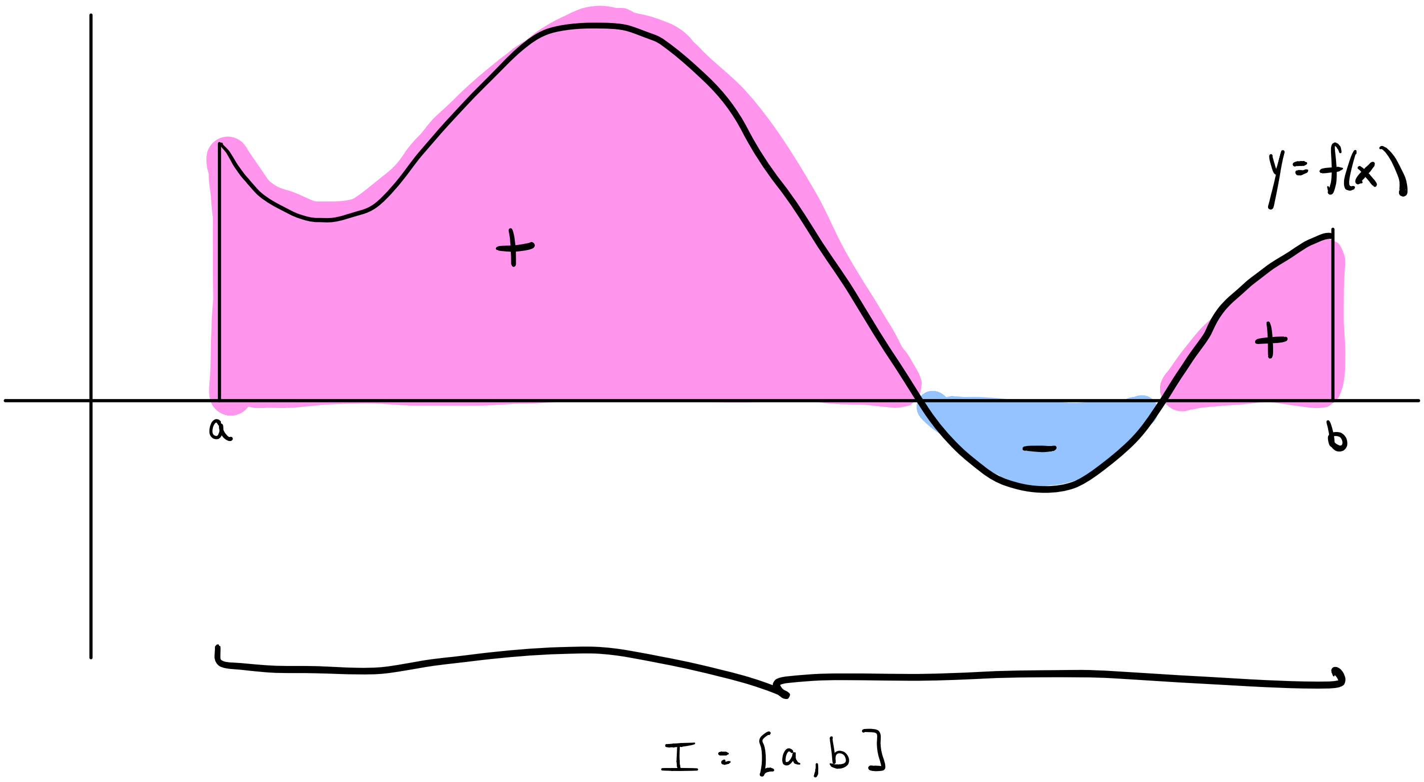


—MAN-KEUNG SIU
University of Hong Kong

$$\left(\frac{1}{3}n(n+1)(n+\tfrac{1}{2}) = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3+3n^2+n}{6}\right)$$

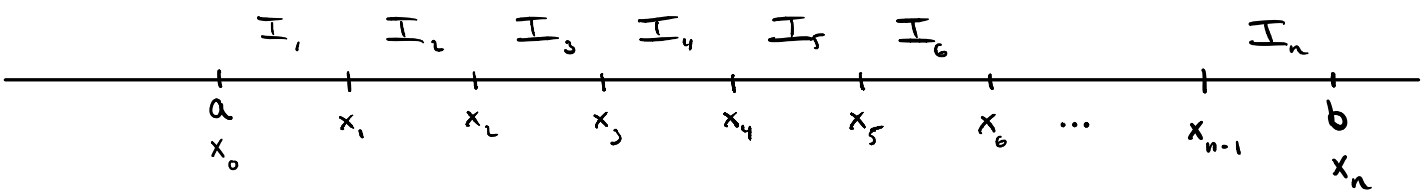
THE PROOF THAT $\lim_{n\to\infty} L_n = \frac{1}{3}$ IS SIMILAR.

MORE GENERALLY, WE CAN CALCULATE THE SIGNED AREA A OF A REGION BOUNDED BY $y = f(x)$, $y = 0$, $x = a$, AND $x = b$.



① SPLIT $I = [a,b]$ INTO n SUBINTERVALS OF EQUAL LENGTH $\Delta x = \frac{b-a}{n}$ WITH ENDPOINTS $a = x_0, x_1, x_2, \ldots, b = x_n$.

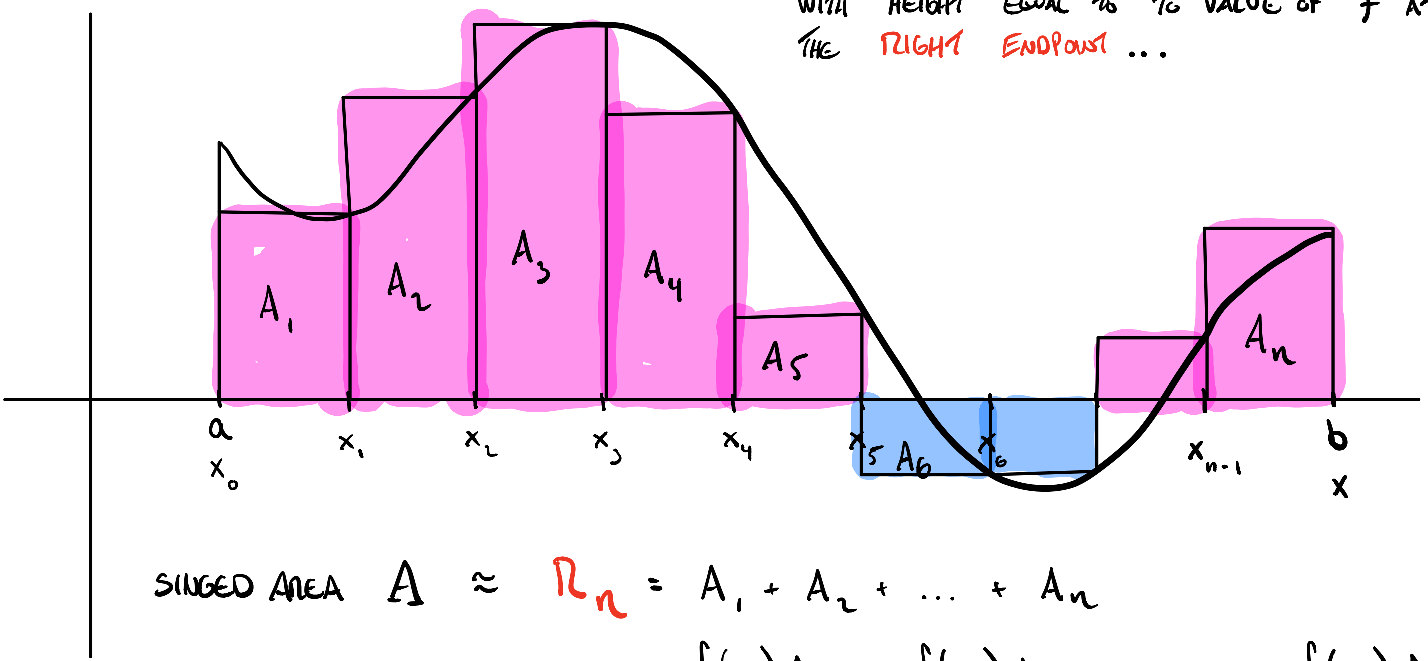
NOTE: $x_i = a + i\Delta x$



NOTE: THE RIGHT ENDPOINT OF SUBINTERVAL I_i IS x_i

THE LEFT ENDPOINT OF SUBINTERVAL I_i IS x_{i-1}

IF WE BUILD RECTANGLES ON EACH SUBINTERVAL WITH HEIGHT EQUAL TO TO VALUE OF f AT THE RIGHT ENDPOINT ...

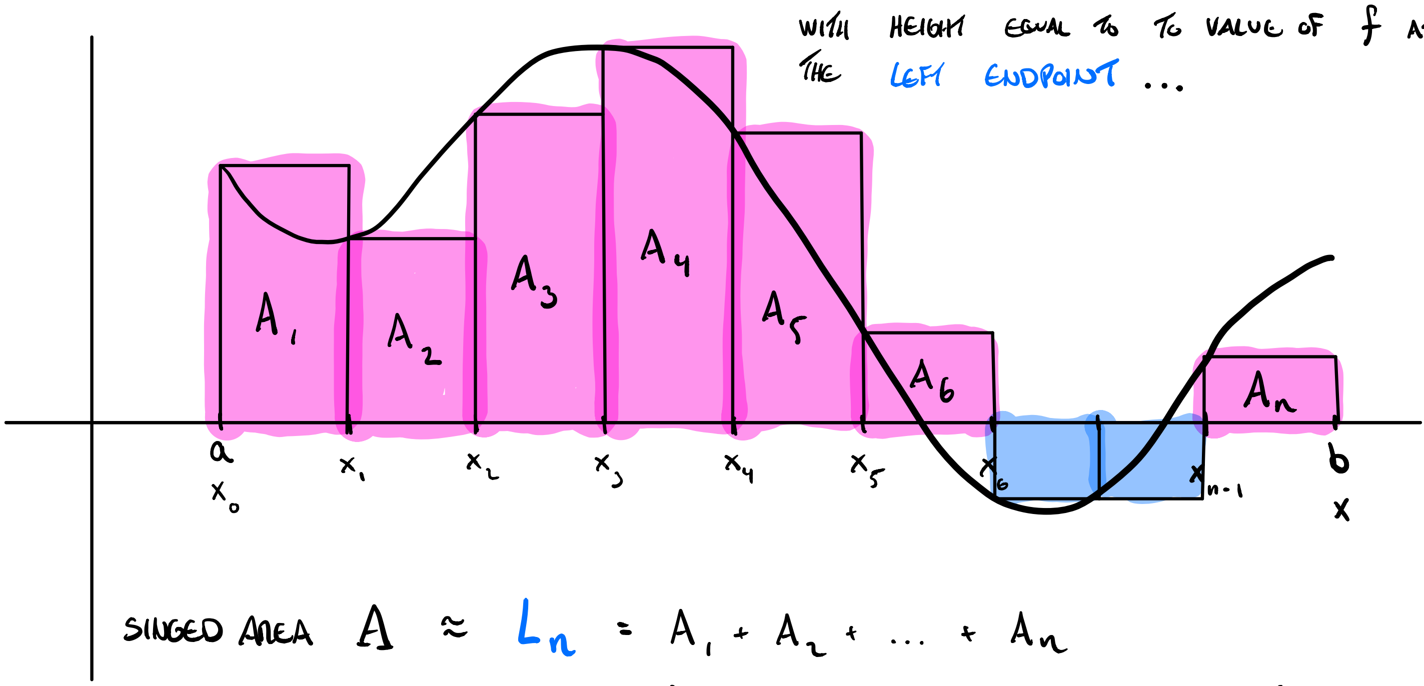


SINGED AREA $A \approx R_n = A_1 + A_2 + \ldots + A_n$

$$= f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x$$

$$= \sum_{i=1}^{n} f(x_i)\Delta x \qquad \Delta x = \frac{b-a}{n}$$

IF WE BUILD RECTANGLES ON EACH SUBINTERVAL WITH HEIGHT EQUAL TO TO VALUE OF f AT THE LEFT ENDPOINT ...



SINGED AREA $A \approx L_n = A_1 + A_2 + \ldots + A_n$

$$= f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x$$

$$= \sum_{i=0}^{n-1} f(x_i)\Delta x \qquad \Delta x = \frac{b-a}{n}$$

Sigma Notation:

"SIGMA" GREEK S SUM

END VALUE FOR INDEX

TEMPLATE FOR TERMS TO SUM

INDEX (VARIABLE)

(ONLY EQUALS INTEGERS)

START VALUE FOR INDEX

$$\sum_{i=3}^{6} \frac{2^{i}}{3i+1}$$

$$\frac{2^{3}}{3\cdot 3+1} + \frac{2^{4}}{3\cdot 4+1} + \frac{2^{5}}{3\cdot 5+1} + \frac{2^{6}}{3\cdot 6+1}$$

INDEX INCREASES BY 1

$$= \frac{8}{10} + \frac{16}{13} + \frac{32}{16} + \frac{64}{19}$$

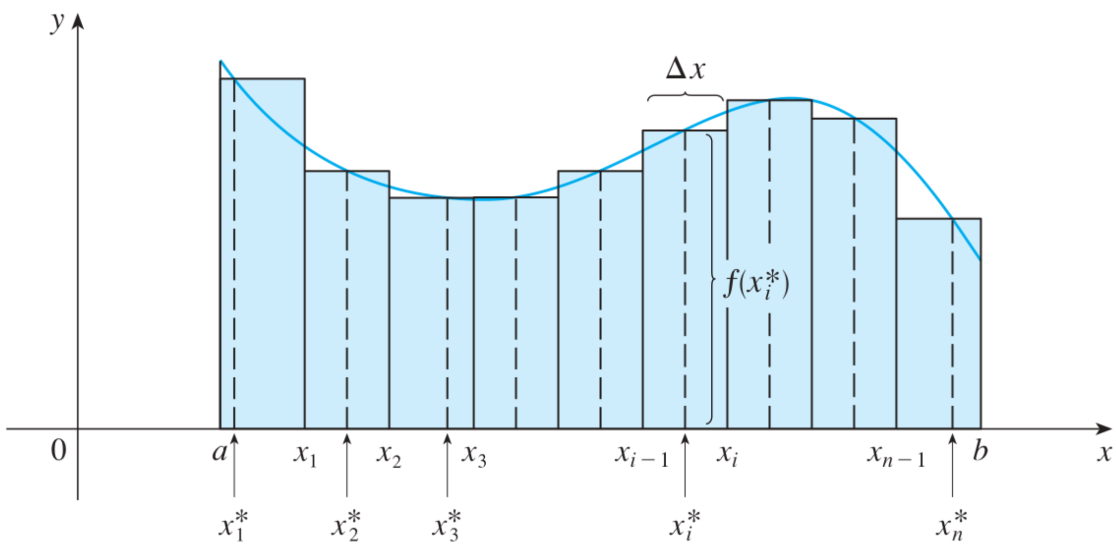
2 Definition The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

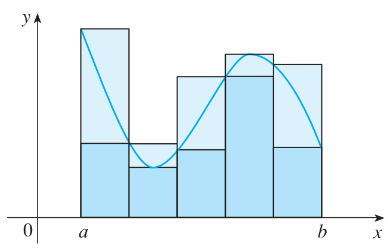
$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x]$$

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_0)\,\Delta x + f(x_1)\,\Delta x + \cdots + f(x_{n-1})\,\Delta x]$$

IN FACT: LET x_i^* BE <u>ANY</u> POINT IN $I_i = [x_{i-1}, x_i]$ (SAMPLE POINT)

$$A = \lim_{n\to\infty} [f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x]$$





← SPECIAL CASES:

UPPER SUM: x_i^* = ABS MAX OF f OVER $[x_{i-1}, x_i]$

LOWER SUM: x_i^* = ABS MIN OF f OVER $[x_{i-1}, x_i]$

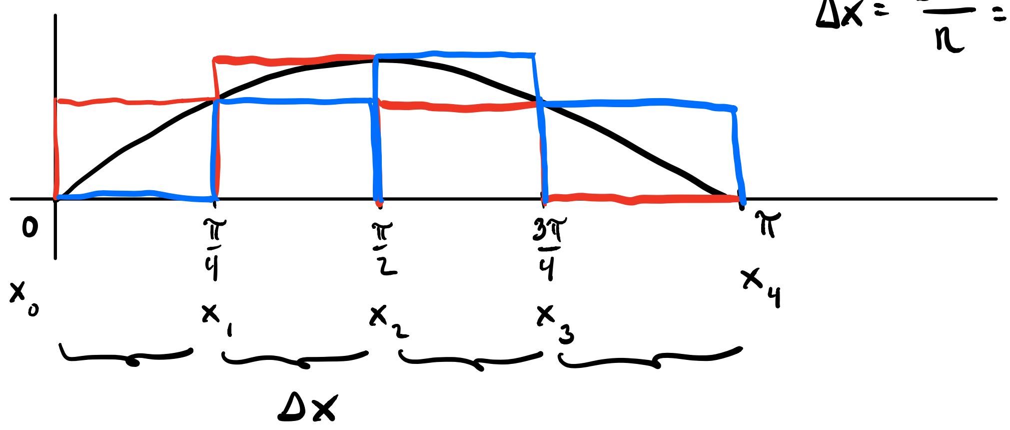
NOTE It can be shown that an equivalent definition of area is the following: *A is the unique number that is smaller than all the upper sums and bigger than all the lower sums.*

ex. ESTIMATE THE AREA UNDER $y = \sin x$ & ABOVE $y = 0$ BETWEEN $x = 0$ & $x = \pi$ USING (a) R_4

(b) L_4

(c) GIVE THE EXACT AREA AS A LIMIT.

$$\Delta x = \frac{b-a}{n} = \frac{\pi - 0}{4} = \frac{\pi}{4}$$



(a) $R_4 = \sum_{i=1}^{4} f(x_i)\Delta x$

$$R_4 = f\left(\tfrac{\pi}{4}\right)\tfrac{\pi}{4} + f\left(\tfrac{\pi}{2}\right)\tfrac{\pi}{4} + f\left(\tfrac{3\pi}{4}\right)\tfrac{\pi}{4} + f(\pi)\tfrac{\pi}{4}$$

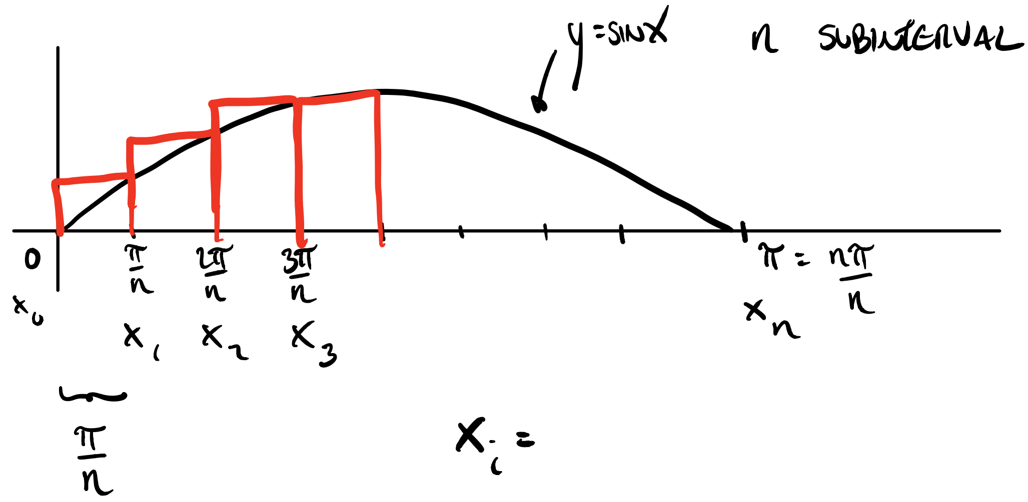
$$R_4 = \frac{\pi}{4}\left(\sin\frac{\pi}{4} + \sin\frac{\pi}{2} + \sin\frac{3\pi}{4} + \sin\pi\right)$$

$$R_4 = \frac{\pi}{4}\left(\frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2} + 0\right) = \frac{\pi(\sqrt{2}+1)}{4}$$

(b) $L_4 = \sum_{i=0}^{n-1} f(x_i)\Delta x$

$$= \frac{\pi}{4}\left(0 + \frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2}\right) = \frac{\pi(\sqrt{2}+1)}{4}$$

(c)



$x_i =$

$$\text{AREA} = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \sin(x_i)\,\frac{\pi}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(\frac{i\pi}{n}\right)\frac{\pi}{n}$$

EXAMPLE 4 Suppose the odometer on our car is broken and we want to estimate the distance driven over a 30-second time interval. We take speedometer readings every five seconds and record them in the following table:

Velocity: 31 35 43 47 45 41

25 31 35 43 47 45

Time (s)	0	5	10	15	20	25	30
Velocity (ft/s)	25	31	35	43	47	45	41

$$\text{VELOCITY} = \frac{\text{DISTANCE}}{\text{TIME}} \implies \text{DISTANCE} = \text{VELOCITY} \times \text{TIME}$$

APPROX. DISTANCE TRAVELED FROM $t=0$ TO $t=5$: $31\ ft/s \times 5\ s = 155\ ft$

APPROX. DISTANCE TRAVELED FROM $t=5$ TO $t=10$: $35\ ft/s \times 5\ s = 175\ ft$

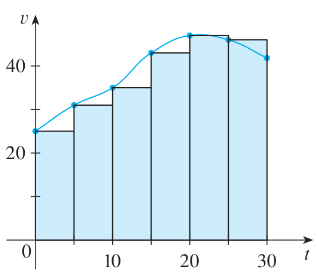
APPROX. DISTANCE TRAVELED FROM $t=10$ TO $t=15$: $43\ ft/s \times 5\ s = 215\ ft$

APPROX. DISTANCE TRAVELED FROM $t=15$ TO $t=20$: $47\ ft/s \times 5\ s = 235\ ft$

APPROX. DISTANCE TRAVELED FROM $t=20$ TO $t=25$: $45\ ft/s \times 5\ s = 225\ ft$

APPROX. DISTANCE TRAVELED FROM $t=25$ TO $t=30$: $41\ ft/s \times 5\ s = 205\ ft$

TOTAL 1210 ft



§4.2 The Definite Integral

2 Definition of a Definite Integral If f is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0 \,(= a), x_1, x_2, \ldots, x_n \,(= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

Anatomy:

Integral sign "S" for limit of Sums

Upper limit/bound

Lower limit/bound

$$\int_a^b f(x)\,dx$$

Differential
For now: bookend that specifies the independent variable

Integrand

$$= \underbrace{\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x}_{\text{Riemann Sum}}$$

Riemann
Bernhard Riemann received his Ph.D. under the direction of the legendary Gauss at the University of Göttingen and remained there to teach. Gauss, who was not in the habit of praising other mathematicians, spoke of Riemann's "creative, active, truly mathematical mind and gloriously fertile originality." The definition (2) of an integral that we use is due to Riemann. He also made major contributions to the theory of functions of a complex variable, mathematical physics, number theory, and the foundations of geometry. Riemann's broad concept of space and geometry turned out to be the right setting, 50 years later, for Einstein's general relativity theory. Riemann's health was poor throughout his life, and he died of tuberculosis at the age of 39.

The precise meaning of the limit that defines the integral is as follows:

For every number $\varepsilon > 0$ there is an integer N such that

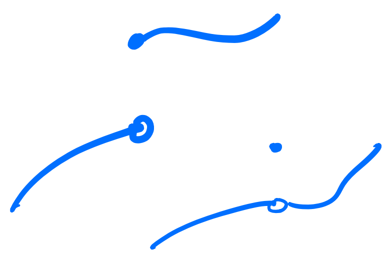
$$\left| \int_a^b f(x)\,dx - \sum_{i=1}^{n} f(x_i^*)\,\Delta x \right| < \varepsilon$$

for every integer $n > N$ and for every choice of x_i^* in $[x_{i-1}, x_i]$.

Note: $\int_a^b f(x)\,dx$ IS A LIMIT.

$\underbrace{\text{IF IT EXISTS}}$, IT IS A REAL NUMBER.

Def: f IS INTEGRABLE IF $\int_a^b f(x)\,dx$ EXIST FOR ALL $a, b \in \mathbb{R}$ SUCH THAT $[a,b] \subseteq \text{Dom}(f)$.



OR REMOVABLE

3 Theorem If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists.

If f is integrable on $[a, b]$, then the limit in Definition 2 exists and gives the same value no matter how we choose the sample points x_i^*. To simplify the calculation of the integral we often take the sample points to be right endpoints. Then $x_i^* = x_i$ and the definition of an integral simplifies as follows.

https://www.geogebra.org/m/CfwjsmHx

4 Theorem If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $\qquad \Delta x = \dfrac{b - a}{n} \qquad$ and $\qquad x_i = a + i\,\Delta x$