Question Details

A particle is moving with the given data. Find the position of the particle.

$a(t) = \cancel{1}12 \sin(t) + 7\cos(t), \quad s(0) = 0, \quad s(2\pi) = \cancel{16}12$

$s(t) =$ $\boxed{\frac{8t}{\pi} - 17\sin(t) - 7\cos(t) + 7}$

$v'(t) = a(t)$ ANTI-DERIV. , $s'(t) = v(t)$ ANTI-DERIV.

$v(t) = -12\cos t + 7\sin t + C$

$s(t) = -12\sin t - 7\cos t + Ct + D$

USING $s(0)=0$ & $s(2\pi)=12$, SOLVE FOR C & D.

$s(0) = -12\sin 0 - 7\cos 0 + C(0) + D = 0$

$0 - 7 + 0 + D = 0 \Rightarrow D = 7$

$s(t) = -12\sin t - 7\cos t + Ct + 7$

$s(2\pi) = -12\sin(2\pi) - 7\cos(2\pi) + C(2\pi) + 7 = 12$

$0 - 7 + 2\pi C + 7 = 12$

$2\pi C = 12$

$C = \frac{12}{2\pi} = \frac{6}{\pi}$

$s(t) = -12\sin t - 7\cos t + \frac{6}{\pi}t + 7$

§ 4.2 The Definite Integral

2 Definition of a Definite Integral If f is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0 \,(= a), x_1, x_2, \ldots, x_n \,(= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

Anatomy:

Integral sign "S" for limit of sums

Upper limit/bound

$$\int_a^b f(x)\,dx$$

Differential
For now: bookend that specifies the independent variable

Lower limit/bound

Integrand

$$= \lim_{n \to \infty} \underbrace{\sum_{i=1}^{n} f(x_i^*)\,\Delta x}_{\text{Riemann Sum}}$$

Riemann

Bernhard Riemann received his Ph.D. under the direction of the legendary Gauss at the University of Göttingen and remained there to teach. Gauss, who was not in the habit of praising other mathematicians, spoke of Riemann's "creative, active, truly mathematical mind and gloriously fertile originality." The definition (2) of an integral that we use is due to Riemann. He also made major contributions to the theory of functions of a complex variable, mathematical physics, number theory, and the foundations of geometry. Riemann's broad concept of space and geometry turned out to be the right setting, 50 years later, for Einstein's general relativity theory. Riemann's health was poor throughout his life, and he died of tuberculosis at the age of 39.

The precise meaning of the limit that defines the integral is as follows:

For every number $\varepsilon > 0$ there is an integer N such that

$$\left| \int_a^b f(x)\,dx - \sum_{i=1}^{n} f(x_i^*)\,\Delta x \right| < \varepsilon$$

for every integer $n > N$ and for every choice of x_i^* in $[x_{i-1}, x_i]$.

NOTE: $\int_a^b f(x)\,dx$ IS A LIMIT.

IF IT EXISTS, IT IS A REAL NUMBER.

DEF: f IS INTEGRABLE IF $\int_a^b f(x)\,dx$ EXIST

FOR ALL $a, b \in \mathbb{R}$ SUCH THAT $[a, b] \subseteq \mathrm{Dom}(f)$.

3 Theorem If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists.

If f is integrable on $[a, b]$, then the limit in Definition 2 exists and gives the same value no matter how we choose the sample points x_i^*. To simplify the calculation of the integral we often take the sample points to be right endpoints. Then $x_i^* = x_i$ and the definition of an integral simplifies as follows.

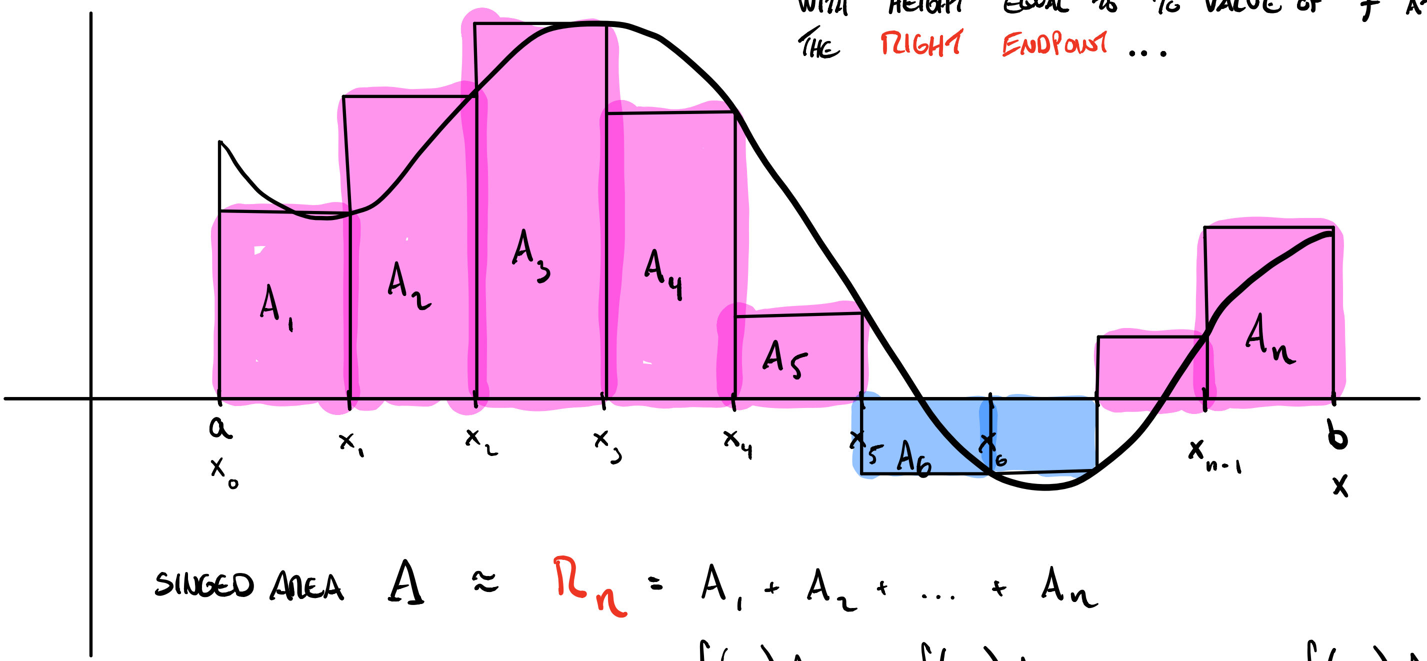
https://www.geogebra.org/m/CfwjsmHx

4 Theorem If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $\qquad \Delta x = \dfrac{b - a}{n} \qquad$ and $\qquad x_i = a + i\,\Delta x$

IF WE BUILD RECTANGLES ON EACH SUBINTERVAL WITH HEIGHT EQUAL TO TO VALUE OF f AT THE RIGHT ENDPOINT ...



SINGED AREA $A \approx R_n = A_1 + A_2 + \dots + A_n$

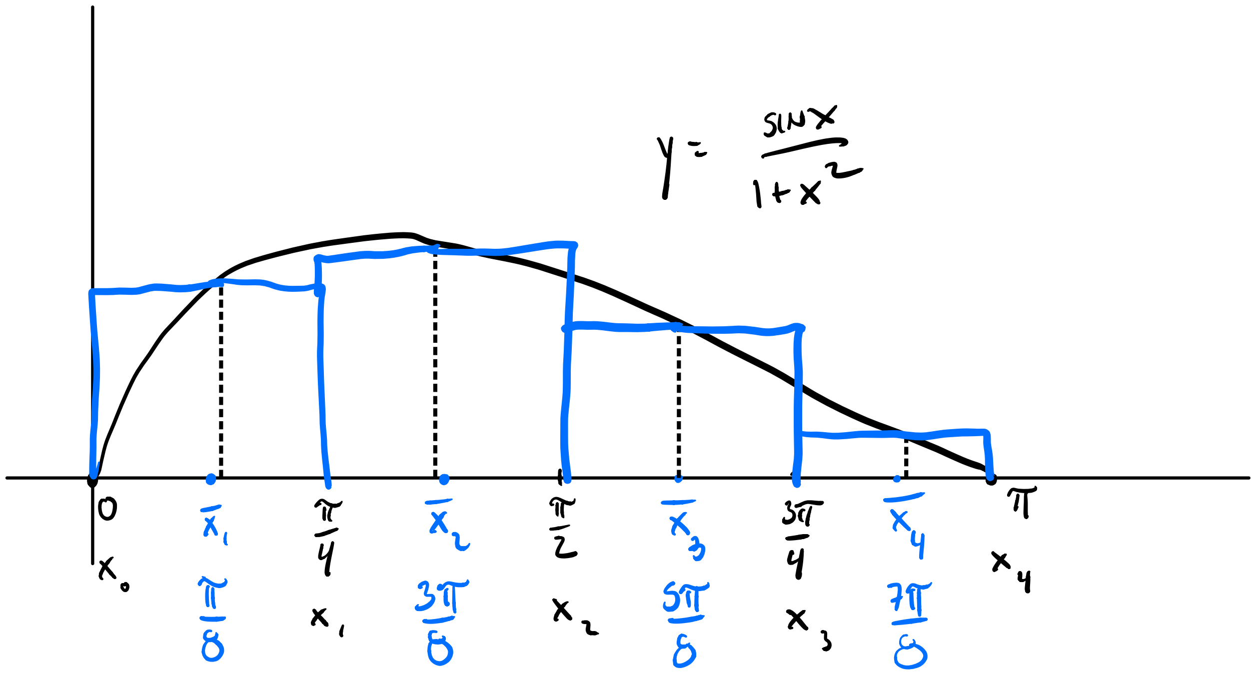
$$= f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x$$

$$= \sum_{i=1}^{n} f(x_i)\Delta x$$

ex. WRITE THE DEFINITE INTEGRAL AS A LIMIT OF RIEMANN SUMS:

$$\int_0^{\pi} \frac{\sin x}{1+x^2}\,dx$$

FIRST APPROXIMATE AS RIEMANN SUM USING MIDPOINTS AND $n=4$ SUBINTERVALS.



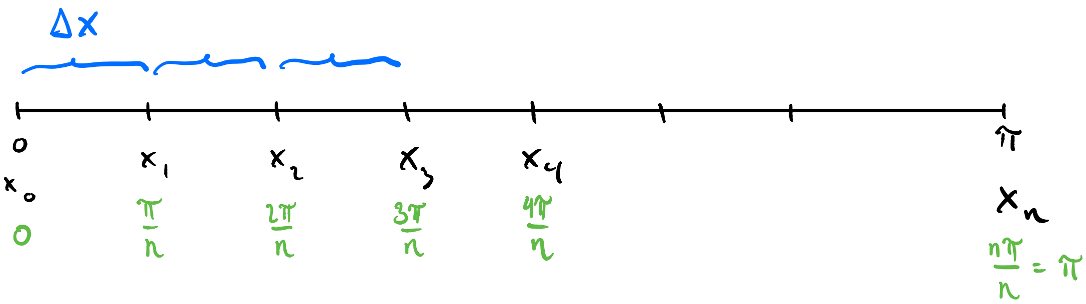
MIDPOINT OF $I_1 = [x_0, x_1]$

$$\int_0^{\pi} \frac{\sin x}{x^2+1}\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\Delta x$$

$$= \frac{\sin\left(\frac{\pi}{8}\right)}{\left(\frac{\pi}{8}\right)^2+1}\left(\frac{\pi}{4}\right) + \frac{\sin\left(\frac{3\pi}{8}\right)}{\left(\frac{3\pi}{8}\right)^2+1}\left(\frac{\pi}{4}\right)$$

$$+ \frac{\sin\left(\frac{5\pi}{8}\right)}{\left(\frac{5\pi}{8}\right)^2+1}\left(\frac{\pi}{4}\right) + \frac{\sin\left(\frac{7\pi}{8}\right)}{\left(\frac{7\pi}{8}\right)^2+1}\left(\frac{\pi}{4}\right)$$

$$\int_0^{\pi} \frac{\sin x}{x^2+1}\,dx$$



$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$f(x) = \frac{\sin x}{x^2+1}$$

$$\Delta x = \frac{b-a}{n} = \frac{\pi - 0}{n} = \frac{\pi}{n}$$

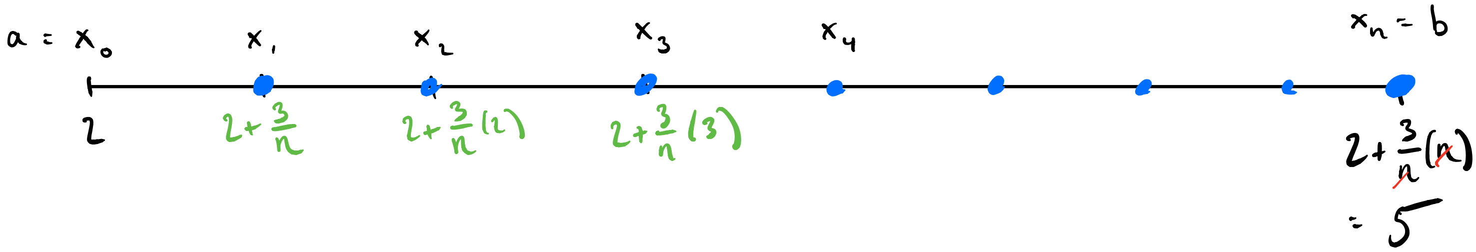
$$x_i = a + i\Delta x = 0 + i\frac{\pi}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\sin\left(\frac{\pi i}{n}\right)}{\left(\frac{\pi i}{n}\right)^2 + 1} \cdot \frac{\pi}{n}$$

ex. EXPRESS THE LIMIT AS A DEFINITE INTEGRAL

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\sqrt{\left(2 + \frac{3i}{n}\right)} \sec^2\left(2 + \frac{3i}{n}\right) \right) \left(\frac{3}{n} \right)$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$



Let $f(x) = \sqrt{x}\sec^2(x)$, $x_i = 2 + \frac{3i}{n}$

$$\Rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\left(\frac{3}{n}\right) \qquad \Delta x = \frac{b-a}{n} = \frac{5-2}{n} = \frac{3}{n}$$

$$\int_2^5 \sqrt{x}\sec^2(x)\,dx$$

5 $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

6 $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

7 $$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

The remaining formulas are simple rules for working with sigma notation:

8 $$\sum_{i=1}^{n} c = nc$$

$$\underbrace{c+c+c+\cdots+c}_{n \text{ TIMES}} = nc$$

9 $$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

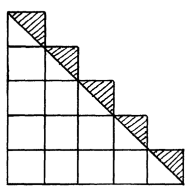
10 $$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

11 $$\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

Proof without words:
Sum of integers

$$1 + 2 + 3 + \cdots + n = \frac{n^2}{2} + \frac{n}{2} = \frac{n(n+1)}{2}$$

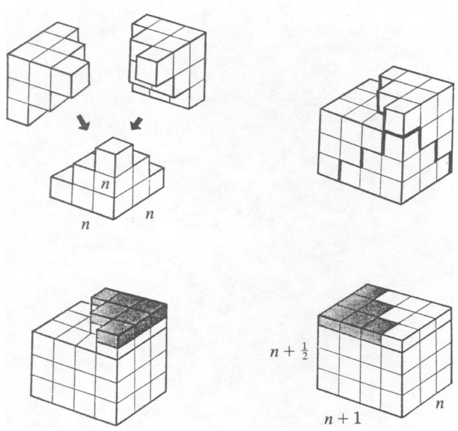
$n=5$



—Ian Richards
University of Minnesota
Minneapolis, MN 55455

Proof without words:
Sum of squares

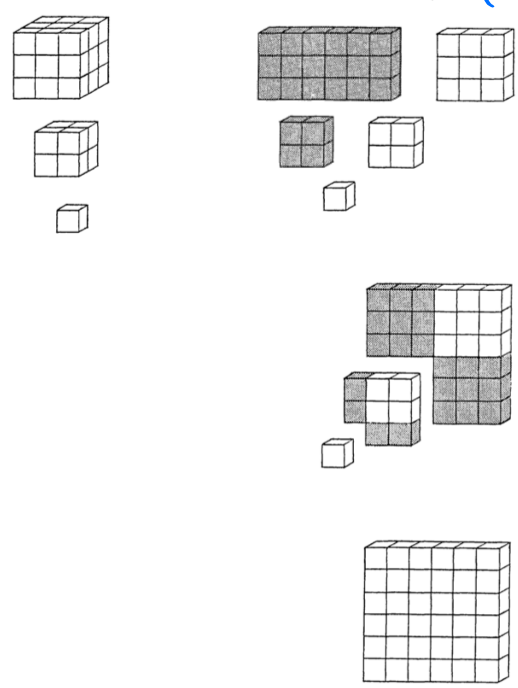
$$1^2 + 2^2 + \cdots + n^2 = \tfrac{1}{3}n(n+1)(n+\tfrac{1}{2}) = \frac{n(n+1)(2n+1)}{6}$$



—Man-Keung Siu
University of Hong Kong

Proof without words:
Sum of cubes

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2 = \left(\frac{n(n+1)}{2}\right)^2$$



—Alan L. Fry
University of North Carolina
at Greensboro

ex. EVALUATE $\displaystyle\int_0^4 5x^3 - 2x \; dx$

$f(x) = 5x^3 - 2x$

$a = 0 \;, \; b = 4$

$\Delta x = \frac{b-a}{n} = \frac{4-0}{n} = \frac{4}{n}$

$x_i = a + i\Delta x = 0 + i \cdot \frac{4}{n} = \frac{4}{n} i$

$$= \lim_{n\to\infty} \left[\sum_{i=1}^{n} f(x_i)\,\Delta x \right]$$

$$= \lim_{n\to\infty} \left[\sum_{i=1}^{n} \left(5\left(\frac{4}{n} i\right)^3 - 2\left(\frac{4}{n} i\right)\right)\left(\frac{4}{n}\right) \right]$$

$$= \lim_{n\to\infty} \left[\sum_{i=1}^{n} \left(\frac{5\cdot 4^4}{n^4} i^3 - \frac{2\cdot 4^2}{n^2} i \right) \right]$$

$$= \lim_{n\to\infty} \left[\frac{5\cdot 4^4}{n^4} \sum_{i=1}^{n} i^3 - \frac{2\cdot 4^2}{n^2} \sum_{i=1}^{n} i \right]$$

$$= \lim_{n\to\infty} \left[\frac{5\cdot 4^4}{n^4} \left(\frac{n(n+1)}{2}\right)^2 - \frac{2\cdot 4^2}{n^2} \left(\frac{n(n+1)}{2}\right) \right]$$

$$= \lim_{n\to\infty} \frac{5\cdot 4^4}{2^2} \left(\frac{n^4 + 2n^3 + n^2}{n^4} \right) - \lim_{n\to\infty} 2\cdot 4^2 \left(\frac{n^2 + n}{n^2} \right)$$

$$= \lim_{n\to\infty} 5\cdot 4^3 \left(\underbrace{\frac{n^4}{n^4}}_{1} + \underbrace{\frac{2n^3}{n^4}}_{0} + \underbrace{\frac{n^2}{n^4}}_{0} \right) - \lim_{n\to\infty} 2\cdot 4^2 \left(\underbrace{\frac{n^2}{n^2}}_{1} + \underbrace{\frac{n}{n^2}}_{0} \right)$$

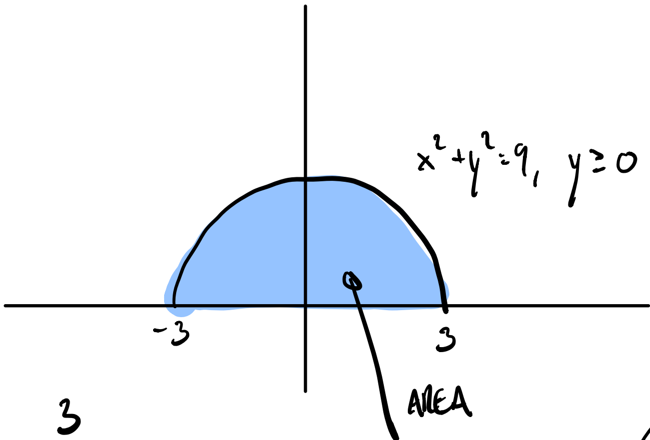
$$= 5\cdot 4^3 - 2\cdot 4^2 = 4^2(5\cdot 4 - 2) = 16(18) = 304$$

ex. EVALUATE $\int_{-3}^{3} \sqrt{9-x^2}\, dx$ = SIGNED AREA BETWEEN X-AXIS & $y=\sqrt{9-x^2}$, $-3 \le x \le 3$

$y = \sqrt{9-x^2}$, $y \ge 0$

$y^2 = 9 - x^2$

$x^2 + y^2 = 9$, $y \ge 0$



$$\int_{-3}^{3} \sqrt{9-x^2}\, dx = \frac{1}{2}\left(\pi \cdot 3^2\right) = \frac{9\pi}{2}$$

ex. Prove that $\displaystyle\int_a^b x^2\,dx = \frac{b^3-a^3}{3}$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(a + \frac{(b-a)}{n} i\right)^2 \left(\frac{b-a}{n}\right)$$

$$= \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} \left(a^2 + \frac{2a(b-a)}{n} i + \left(\frac{b-a}{n}\right)^2 i^2\right)$$

$$= \lim_{n\to\infty} \left[\frac{a^2(b-a)}{n} \sum_{i=1}^{n} 1 + \frac{2a(b-a)^2}{n^2} \sum_{i=1}^{\infty} i + \left(\frac{b-a}{n}\right)^3 \sum_{i=1}^{n} i^2 \right]$$

$$= \lim_{n\to\infty} \left[\frac{a^2(b-a)}{n} n + \frac{2a(b-a)^2}{n^2} \cdot \frac{n(n+1)}{2} + \frac{(b-a)^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \right]$$

$$= \lim_{n\to\infty} \left[a^2(b-a) + a(b-a)^2\left(1 + \frac{1}{n}\right) + \frac{(b-a)^3}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) \right]$$

$$= a^2(b-a) + a(b-a)^2 + \frac{(b-a)^3}{3}$$

$$b^3 + \cancel{ab^2} + \cancel{a^2b} - \cancel{ab^2} - \cancel{a^2b} - a^3$$

$$= \frac{3a^2(b-a) + 3a(b-a)^2 + (b-a)(b^2-2ab+a^2)}{3}$$

$$= \frac{(b-a)\left[3a^2 + 3a(b-a) + b^2 - 2ab + a^2\right]}{3}$$

$$= \frac{(b-a)\left[3a^2 + 3ab - 3a^2 + b^2 - 2ab + a^2\right]}{3}$$

$$= \frac{(b-a)\left[b^2 + ab + a^2\right]}{3} = \frac{b^3-a^3}{3}$$

Properties of the Definite Integral

When we defined the definite integral $\int_a^b f(x)\,dx$, we implicitly assumed that $a < b$. But the definition as a limit of Riemann sums makes sense even if $a > b$. Notice that if we reverse a and b, then Δx changes from $(b - a)/n$ to $(a - b)/n$. Therefore

$$\boxed{\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx}$$

If $a = b$, then $\Delta x = 0$ and so

$$\boxed{\int_a^a f(x)\,dx = 0}$$

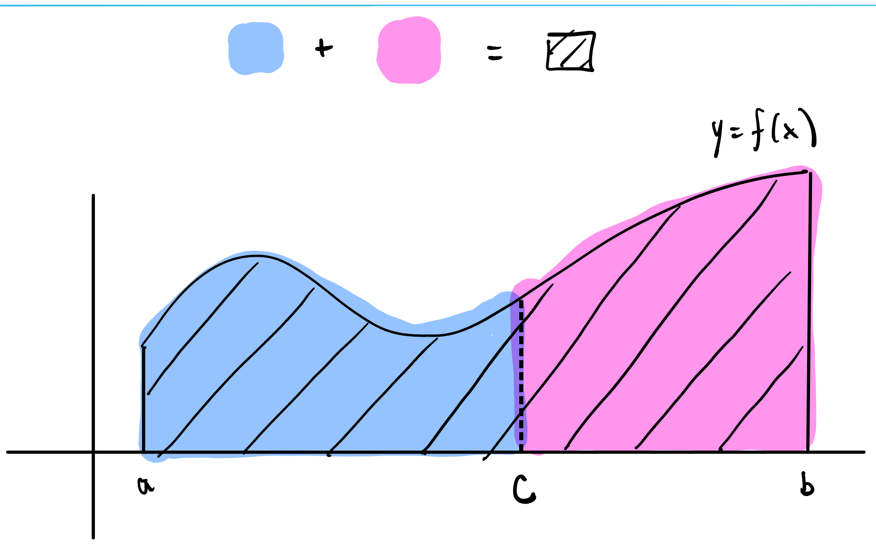
We now develop some basic properties of integrals that will help us to evaluate integrals in a simple manner. We assume that f and g are continuous functions.

Properties of the Integral

1. $\int_a^b c\,dx = c(b - a)$, where c is any constant $\boxed{8}$
2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$ $\boxed{10}$
3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant $\boxed{9}$
4. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$ $\boxed{11}$

CORRESPONDING PROPERTIES OF SUMS (ABOVE)

5. $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$



$a < c < b$

↑

NOT NECESSARY!

ex. IF $\int_3^{12} f(x) = -10$ & $\int_8^{12} 2f(x) = 4$, FIND $\int_3^8 3f(x)\,dx$.

$$\int_3^8 f(x)\,dx + \int_8^{12} f(x) = \int_3^{12} f(x)$$

$$\int_3^8 f(x)\,dx = \int_3^{12} f(x) - \int_8^{12} f(x) = -10 - 4 = -14$$

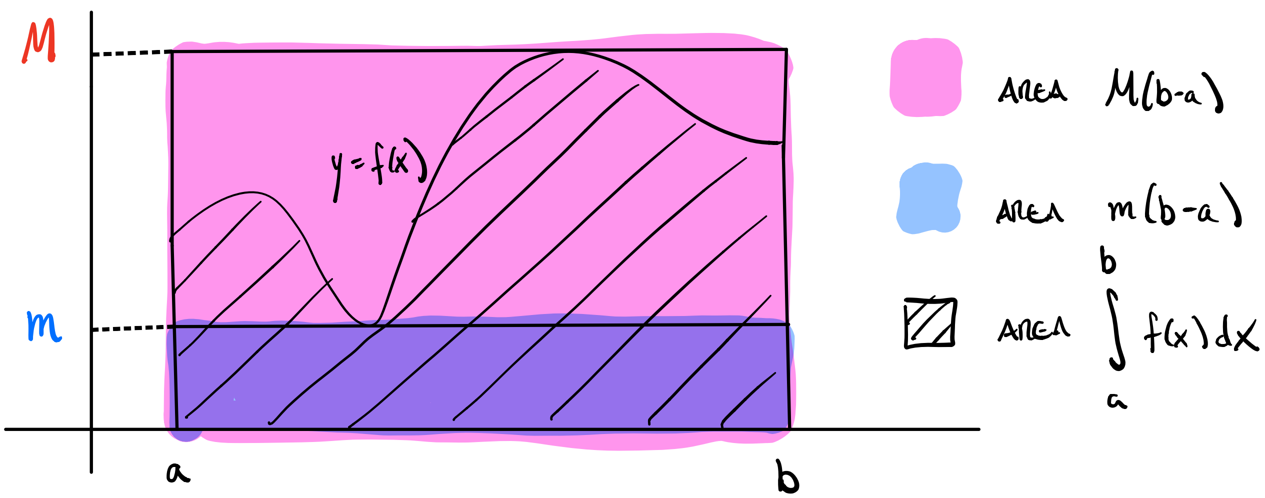
Comparison Properties of the Integral

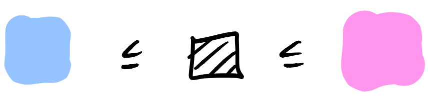
6. If $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant 0$.

7. If $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$.

8. If $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, then

$$m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a)$$





ex. USE PROPERTY 8 TO ESTIMATE $\int_{\pi}^{2\pi} x - 2\sin x\,dx$

$$m(2\pi - \pi) \leq \int_{\pi}^{2\pi} x - 2\sin x\,dx \leq M(2\pi - \pi)$$

m IS ABS MIN VALUE OF f OVER $[\pi, 2\pi]$,

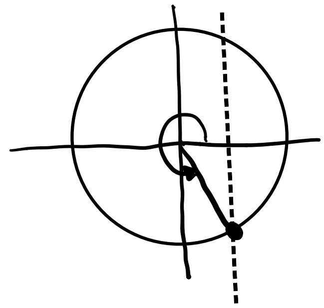
M IS ABS MAX VALUE OF f OVER $[\pi, 2\pi]$.

CRIT. PNTS: $f(x) = x - 2\sin x$

$f'(x) = 1 - 2\cos x = 0$

$\cos x = \frac{1}{2}$, $\underline{\pi \le x \le 2\pi}$

$x = \frac{5\pi}{3}$



$x = \frac{1}{2}$

x	$f(x) = x - 2\sin(5\pi/3)$	
π	π	MIN
$5\pi/3$	$5\pi/3 + \sqrt{3}/2$	MAX
2π	2π	

$$\pi^2 \le \int_{\pi}^{2\pi} x - 2\sin x \, dx \le \left(\frac{5\pi}{3} + \frac{\sqrt{3}}{2}\right)\pi$$

§ 4.3 The Fundamental Theorem of Calculus

We can use the definite integral to define functions.

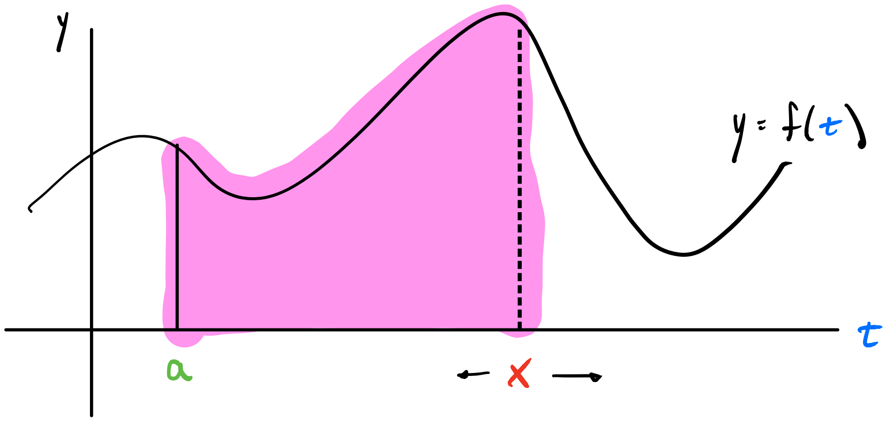
Consider the "area so far" function

Place-holder

$$g(x) = \int_a^x f(t)\,dt$$

$\updownarrow$

Area below $y = f(t)$
$a \le t \le x$



* Note you can't have the same variable appear in integrand & limits of integration!

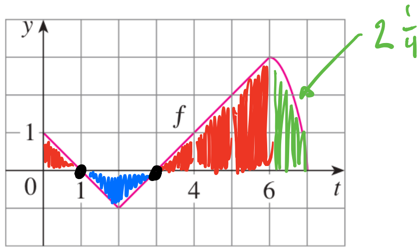
2. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.

(a) Evaluate $g(x)$ for $x = 0, 1, 2, 3, 4, 5,$ and 6.

(b) Estimate $g(7)$.

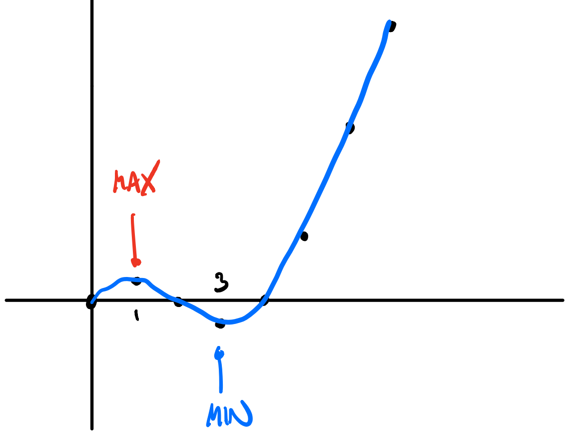
(c) Where does g have a maximum value? Where does it have a minimum value?

(d) Sketch a rough graph of g.



x	$g(x) = \int_0^x f(t)\,dt$
0	0
1	1/2
2	0
3	-1/2
4	0
5	3/2
6	4
7	≈ 6 1/4

1 3



$f(x) = 0$ when

$g(x)$ = local max/min value.