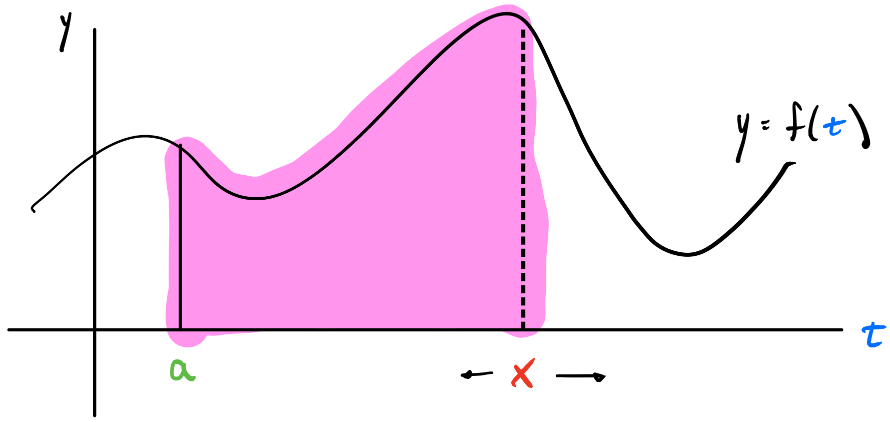
§4.3 The Fundamental Theorem of Calculus

We can use the definite integral to define functions.

Consider the "area so far" function

$$g(x) = \int_a^x f(t)\,dt$$



* Note you can't have the same variable appear in integrand & limits of integration!

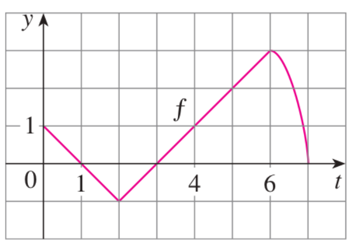
2. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.

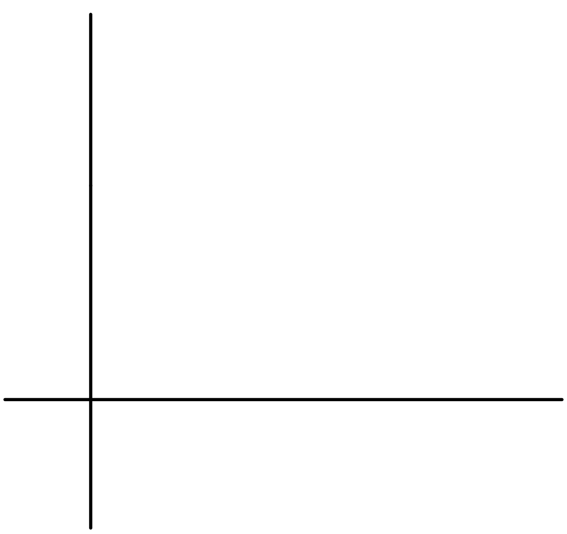
(a) Evaluate $g(x)$ for $x = 0, 1, 2, 3, 4, 5$, and 6.

(b) Estimate $g(7)$.

(c) Where does g have a maximum value? Where does it have a minimum value?

(d) Sketch a rough graph of g.





x	$g(x) = \int_0^x f(t)\,dt$
0	
1	
2	
3	
4	
5	
6	
7	

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

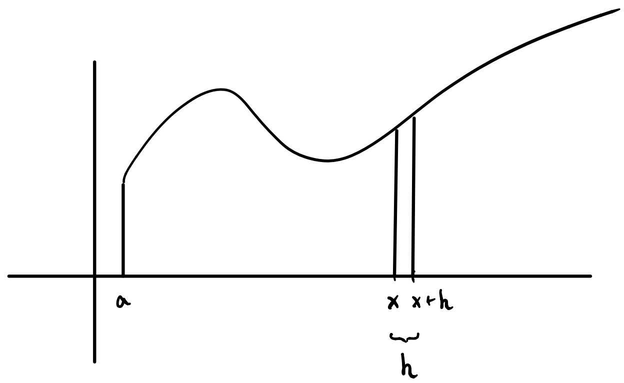
i.e. $$\frac{d}{dx} \int_a^x f(t)\,dt = f(x).$$

PROOF: WE MUST SHOW THAT $g'(x)$ EXISTS FOR ALL $x \in (a,b)$
(CONTINUITY AT $x=a$ & $x=b$ IS NOT HARD TO SEE.)

FOR ANY $x \in (a,b)$, WE HAVE

$$g'(x) = \lim_{h\to 0} \frac{1}{h}\left(g(x+h) - g(x)\right) = \lim_{h\to 0} \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right)$$

$$= \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt$$



EXTREME VALUE THM $\Rightarrow$ $\exists\, u, v \in [x, x+h]$ SUCH THAT

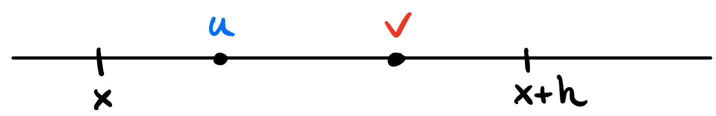
$$f(u) \le f(t) \le f(v) \qquad \forall t \in [x, x+h]$$

$$\Rightarrow \quad f(u)h \le \int_x^{x+h} f(t)\,dt \le f(v)h \quad * \qquad \text{(PROPERTY 8 OF DEF. INT.)}$$

$$\Rightarrow \quad f(u) \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le f(v)$$

* ASSUMING f IS POSITIVE $\forall t \in [x, x+h]$ AND $h > 0$.
IF NOT THEN INEQUALITIES MAY NEED TO BE REVERSED. NO BIG DEAL.

$$\therefore \quad f(u) \le \frac{g(x+h) - g(x)}{h} \le f(v)$$



$x \le u, v \le x+h$

AS $h \to 0$ BOTH $u \to x$
$v \to x$

$$\Rightarrow \quad \lim_{h\to 0} f(u) \le \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \le \lim_{h\to 0} f(v)$$

$$\Rightarrow \quad \lim_{u\to x} f(u) \le \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \le \lim_{v\to x} f(v)$$

$\Rightarrow \quad g'(x) = f(x)$ BY SQUEEZE THM. $\square$

<u>ex.</u> (a) FIND $\displaystyle \frac{d}{dx} \int_{\pi}^{x} \frac{\sec(\pi t)}{\sqrt{t^2+1}}\, dt$

(b) LET $\displaystyle g(x) = \int_{\pi}^{x} \frac{\sec(\pi t)}{\sqrt{t^2+1}}\, dt$

FIND $g'(x)$.

BOTH: $\displaystyle \frac{\sec(\pi x)}{\sqrt{x^2+1}}$

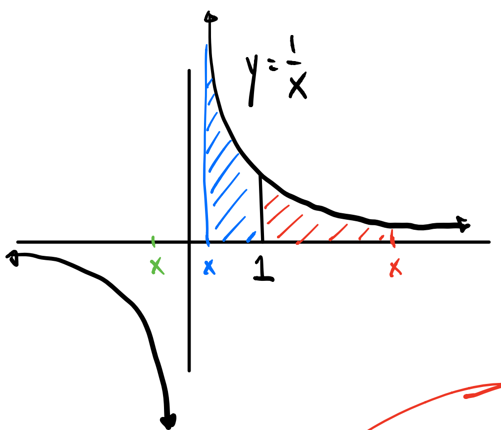
ex. Let $g(x) = \int_x^1 \frac{1}{t}\,dt = -\int_1^x \frac{1}{t}\,dt$

(a) What is the domain of g?

$1 \in Dom(g)$: $g(1) = \int_1^1 \frac{1}{t}\,dt = 0$

$0 \notin Dom(g)$ because $\frac{1}{x}$ is discontinuous at $x=0$

Recall: $\int_a^b f(x)\,dx$ is defined when f is continuous on $[a,b]$.



$\int_a^x f(t)\,dt$ is defined only when f is continuous between a & x.

$$Dom(g) = (0, \infty)$$

e.g. $\int_{-2}^{3} \frac{1}{x}\,dx$ undefined

because $\frac{1}{x}$ has ∞-discontinuity at $x=0$, $-2 < 0 < 3$

($0 \in$ interval of int.)

(b) What is $g'(x)$?

$g(x) = \int_x^1 \frac{1}{t}\,dt = -\int_1^x \frac{1}{t}\,dt$

$g'(x) = \frac{d}{dx}\left[-\int_1^x \frac{1}{t}\,dt\right] = -\frac{1}{x}$

ex. FIND $\displaystyle \frac{d}{dx}\int_0^{\sin(x^2)} (t+3)^2\,dt$

① LET $u = \sin(x^2)$. $\displaystyle \frac{du}{dx} = \cos(x^2)\cdot 2x = 2x\cos(x^2)$

FIND $\displaystyle \frac{d}{dx}\int_0^{u} (t+3)^2\,dt$

② CHAIN RULE : $\displaystyle \frac{d}{du}\left[\int_0^{u} (t+3)^2\,dt\right]\frac{du}{dx}$

$\displaystyle = (u+3)^2\frac{du}{dx}$ (FTC)

$\displaystyle = \boxed{(\sin(x^2)+3)^2\,2x\cos(x^2)}$

FIND $\displaystyle \frac{d}{dx}\int_0^{\sin(x^2)} (t+3)^2\,dt$

① LET $\displaystyle g(x) = \int_0^{x} (t+3)^2\,dt$. $g'(x) = (x+3)^2$

NOTE $\displaystyle g(\sin(x^2)) = \int_0^{\sin(x^2)} (t+3)^2\,dt$

$\displaystyle \therefore \frac{d}{dx}\int_0^{\sin(x^2)} (t+3)^2\,dt = \frac{d}{dx}g(\sin(x^2)) = g'(\sin(x^2))\,2x\cos(x^2)$

$$= \boxed{\left(\sin(x^2)+3\right)^2 2x\cos(x^2)}$$

$$\boxed{\text{IN GENERAL}: \quad \frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))\,g'(x)}$$

PROOF: LET $u = g(x)$. THEN

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = \frac{d}{du}\int_a^{u} f(t)\,dt\ \frac{du}{dx}$$

$$= f(u)\frac{du}{dx} = f(g(x))\,g'(x) \qquad \square$$

<u>NOTE:</u>

$$\frac{d}{dx}\int_{m(x)}^{M(x)} f(t)\,dt = f(M(x))\,M'(x) - f(m(x))\,m'(x).$$

PROOF:

$$\frac{d}{dx}\int_{m(x)}^{M(x)} f(t)\,dt = \frac{d}{dx}\left[\left(\int_{m(x)}^{a}\right) f(t)\,dt + \int_a^{M(x)} f(t)\,dt\right]$$

$$= \frac{d}{dx}\left[\int_a^{M(x)} f(t)\,dt - \int_a^{m(x)} f(t)\,dt\right]$$

$$= f(M(x))\,M'(x) - f(m(x))\,m'(x).$$

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) \; *$$

where F is any antiderivative of f, that is, a function F such that $F' = f$.

$*$ NOTATION: $F(x)\Big|_a^b = F(b) - F(a)$

$F(x)\Big]_a^b = F(b) - F(a)$

$\Big[F(x)\Big]_a^b = F(b) - F(a)$

PROOF:

LET $g(x) = \int_a^x f(t)\,dt$

FTC 1 $\Rightarrow$ g IS AN ANTIDERIVATIVE OF f

$(g'(x) = f(x))$

OBVIOUSLY $\int_a^b f(t)\,dt = g(b) - g(a)$ (WHY?)

$\int_a^b f(t)\,dt - \underbrace{\int_a^a f(t)\,dt}_{0}$ ✓

NOW SUPPOSE F IS ANY ANTIDERIVATIVE OF f

THEN $F(x) = g(x) + C$ FOR SOME C AND

$F(b) - F(a) = g(b) + C - (g(a) + C) = g(b) - g(a) = g(b) = \int_a^b f(x)\,dx$ $\square$

ex. FIND $\displaystyle\int_1^2 \frac{(x^2-1)^2}{x^2}\,dx$

$$= \int_1^2 \frac{x^4 - 2x^2 + 1}{x^2}\,dx = \int_1^2 x^2 - 2 + x^{-2}\,dx$$

$$= \frac{1}{3}x^3 - 2x - x^{-1}\Big|_1^2 \quad *$$

$$= \left(\frac{1}{3}(2)^3 - 2(2) - (2)^{-1}\right) - \left(\frac{1}{3}(1)^3 - 2(1) - (1)^{-1}\right)$$

$$= \frac{8}{3} - 4 - \frac{1}{2} - \frac{1}{3} + 2 + 1 = \frac{5}{6}$$

$$* = \frac{1}{3}\left[x^3\right]_1^2 - 2\left[x\right]_1^2 - \left[x^{-1}\right]_1^2$$

$$= \frac{1}{3}\left(2^3 - 1^2\right) - 2(2-1) - \left(2^{-1} - 1^{-1}\right)$$

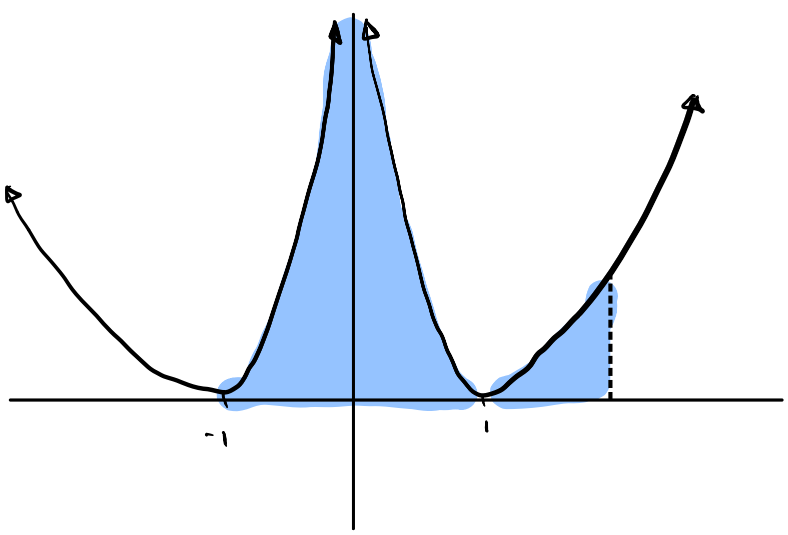
$$= \frac{7}{3} - 2 - \left(-\frac{1}{2}\right) = \frac{5}{6}$$

ex. WHAT IS WRONG WITH THE FOLLOWING ?

$$\int_{-1}^{2} \frac{(x^2-1)^2}{x^2}\,dx = \frac{1}{3}x^3 - 2x - x^{-1}\Big|_{-1}^{2}$$

$$= \frac{1}{3}\left(2^3-(-1)^3\right) - 2\left(2-(-1)\right) - \left(2^{-1}-(-1)^{-1}\right)$$

$$= \frac{1}{3}(9) - 2(3) - \left(\frac{3}{2}\right) = -\frac{9}{2}$$



$x=0$
VERTICAL ASYMPTOTE !

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

ONLY WHEN f IS CONTINUOUS ON $[a,b]$ & $F'(x) = f(x)$ FOR ALL $x \in (a,b)$.

More Practice ?

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7–18 Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of the function.

7. $g(x) = \int_0^x \sqrt{t + t^3}\, dt$

8. $g(x) = \int_1^x \cos(t^2)\, dt$

9. $g(s) = \int_5^s (t - t^2)^8\, dt$

10. $h(u) = \int_0^u \frac{\sqrt{t}}{t + 1}\, dt$

11. $F(x) = \int_x^0 \sqrt{1 + \sec t}\, dt$

$\left[\textit{Hint: } \int_x^0 \sqrt{1 + \sec t}\, dt = -\int_0^x \sqrt{1 + \sec t}\, dt \right]$

12. $R(y) = \int_y^2 t^3 \sin t\, dt$

13. $h(x) = \int_2^{1/x} \sin^4 t\, dt$

14. $h(x) = \int_1^{\sqrt{x}} \frac{z^2}{z^4 + 1}\, dz$

15. $y = \int_1^{3x+2} \frac{t}{1 + t^3}\, dt$

16. $y = \int_0^{x^4} \cos^2\theta\, d\theta$

17. $y = \int_{\sqrt{x}}^{\pi/4} \theta \tan\theta\, d\theta$

18. $y = \int_{\sin x}^1 \sqrt{1 + t^2}\, dt$

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19–38 Evaluate the integral.

19. $\int_1^3 (x^2 + 2x - 4)\, dx$

20. $\int_{-1}^1 x^{100}\, dx$

21. $\int_0^2 \left(\frac{4}{5}t^3 - \frac{3}{4}t^2 + \frac{2}{5}t\right) dt$

22. $\int_0^1 \left(1 - 8v^3 + 16v^7\right) dv$

23. $\int_1^9 \sqrt{x}\, dx$

24. $\int_1^8 x^{-2/3}\, dx$

25. $\int_{\pi/6}^{\pi} \sin\theta\, d\theta$

26. $\int_{-5}^5 \pi\, dx$

27. $\int_0^1 (u + 2)(u - 3)\, du$

28. $\int_0^4 (4 - t)\sqrt{t}\, dt$

29. $\int_1^4 \frac{2 + x^2}{\sqrt{x}}\, dx$

30. $\int_{-1}^2 (3u - 2)(u + 1)\, du$

31. $\int_{\pi/6}^{\pi/2} \csc t \cot t\, dt$

32. $\int_{\pi/4}^{\pi/3} \csc^2\theta\, d\theta$

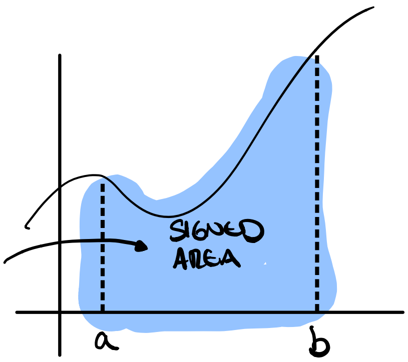
4.4 The Indefinite Integral & the Net Change Thm.

<u>Def</u>: The **indefinite integral** $\int f(x)\,dx$

is used to represent the **entire collection** of antiderivatives of $f(x)$. Thus, if $F'(x) = f(x)$ then

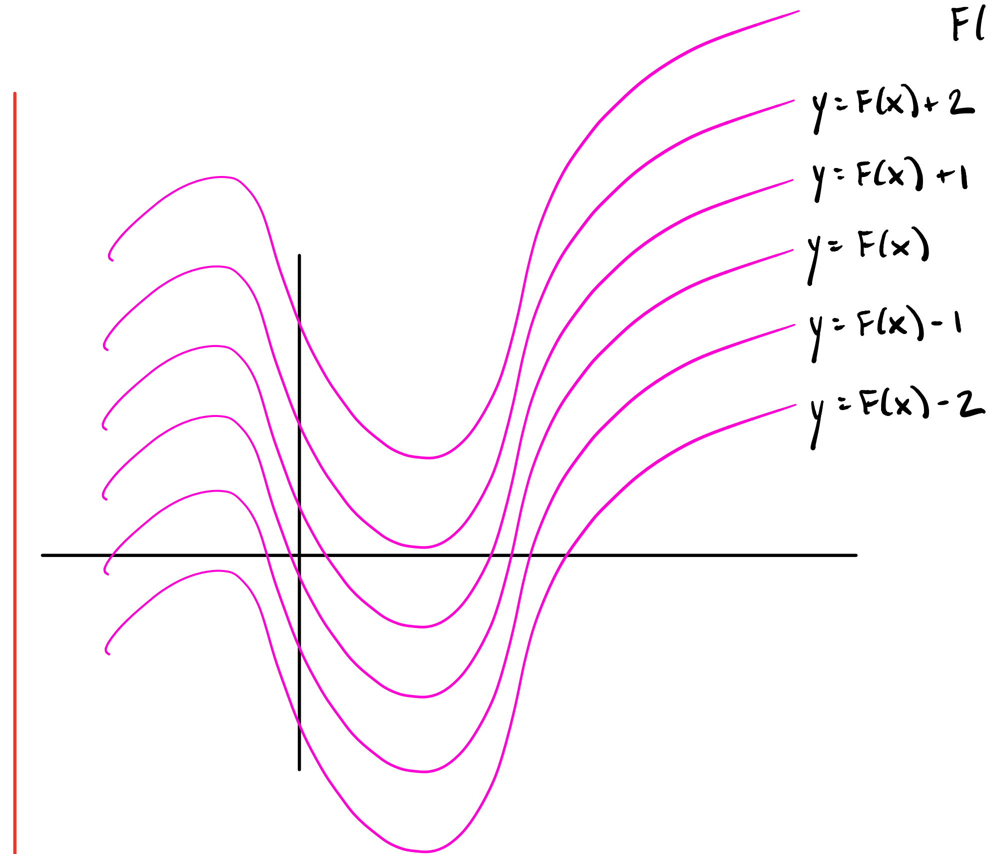
$$\int f(x)\,dx = F(x) + C.$$

Note: Definite integral $\int_a^b f(x)\,dx$ = **number**



Indefinite integral $\int f(x)\,dx$ = **collection of functions**

$F(x) + C$



ex. $\int \sec x \tan x \, dx = \sec x + C$

BECAUSE $\frac{d}{dx}\left[\sec x\right] = \sec x \tan x$.

$$\left(\frac{d}{dx}\left[\frac{1}{\cos x}\right] = \frac{\cos x\,(0) - (-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} \;\checkmark \right)$$

DERIVATIVE RULES $\rightsquigarrow$ INTEGRATION RULES :

$\int cf(x)\,dx = c\int f(x)\,dx$	$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$
$\int k\,dx = kx + C$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
$\int \sin x\,dx = -\cos x + C$	$\int \cos x\,dx = \sin x + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \csc^2 x\,dx = -\cot x + C$
$\int \sec x \tan x\,dx = \sec x + C$	$\int \csc x \cot x\,dx = -\csc x + C$

ex. FIND $\int \frac{x^2 + 3x - 1}{\sqrt{x}} + \sec^2(2x)\, dx$

$$= \int \frac{x^2}{x^{1/2}} + \frac{3x}{x^{1/2}} - \frac{1}{x^{1/2}} + \sec^2(2x)\, dx$$

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C$$

$$= \int x^{3/2} + 3x^{1/2} - x^{-1/2} + \sec^2(2x)\, dx$$

$$= \int x^{3/2} dx + 3\int x^{1/2} dx - \int x^{-1/2} dx + \int \sec^2(2x)\, dx$$

$$= \frac{2}{5} x^{5/2} + C_1 + 3 \frac{1}{\frac{1}{2}+1} x^{\frac{1}{2}+1} + C_2 - \frac{1}{-\frac{1}{2}+1} x^{-\frac{1}{2}+1} + C_3 + \frac{1}{2} \tan(2x) + C_4$$

$$\frac{d}{dx}\left[\frac{\tan(2x)}{2}\right] = \frac{\sec^2(2x) \cdot 2}{2}$$

$$= \frac{2}{5} x^{5/2} + \cancel{3} \cdot \frac{2}{\cancel{3}} x^{3/2} - 2x^{1/2} + \frac{1}{2}\tan(2x) + \underbrace{C_1 + C_2 + C_3 + C_4}_{\text{SUM OF 4 UNKNOWN \#'S}}$$

$$\frac{d}{dx}\left[\frac{1}{2}\tan(2x)\right] = \sec^2(2x)$$

$$\cancel{\tfrac{1}{2}} \sec^2(2x)\, \cancel{2}$$

$$\left(C = C_1 + C_2 + C_3 + C_4 \right)$$

= UNKNOWN #'S

$$\int \frac{x^2 + 3x - 1}{\sqrt{x}} + \sec^2(2x)\, dx = \boxed{\frac{2}{5} x^{5/2} + 2x^{3/2} - 2x^{1/2} + \frac{1}{2}\tan(2x) + C}$$

37. $\int_0^1 \left(\sqrt[4]{x^5} + \sqrt[5]{x^4}\right) dx$ $= F(1) - F(0)$ (FTC II)

WHERE $F'(x) = \sqrt[4]{x^5} + \sqrt[5]{x^4}$

(F IS ANTIDERIV. OF $\sqrt[4]{x^5} + \sqrt[5]{x^4}$)

NOTE: $\sqrt[4]{x^5} + \sqrt[5]{x^4} = (x^5)^{1/4} + (x^4)^{1/5} = x^{5/4} + x^{4/5}$

$$\int_0^1 x^{5/4} + x^{4/5}\, dx = \int_0^1 x^{5/4}\, dx + \int_0^1 x^{4/5}\, dx$$

$$\int x^n\, dx = \frac{1}{n+1} x^{n+1} + C$$

$$= \frac{1}{\frac{5}{4}+1} x^{\frac{5}{4}+1} \Big|_0^1 + \frac{1}{\frac{4}{5}+1} x^{\frac{4}{5}+1} \Big|_0^1$$

$$= \frac{4}{9} x^{9/4} \Big|_0^1 + \frac{5}{9} x^{9/5} \Big|_0^1 \quad \frac{4}{9}(1)^{9/4} + \frac{5}{9}(1)^{9/5} + c - (0 + c)$$

$$= \left[\frac{4}{9} x^{9/4} + \frac{5}{9} x^{9/5} + c \right]_0^1 \quad = \frac{4}{9} + \frac{5}{9} = 1$$

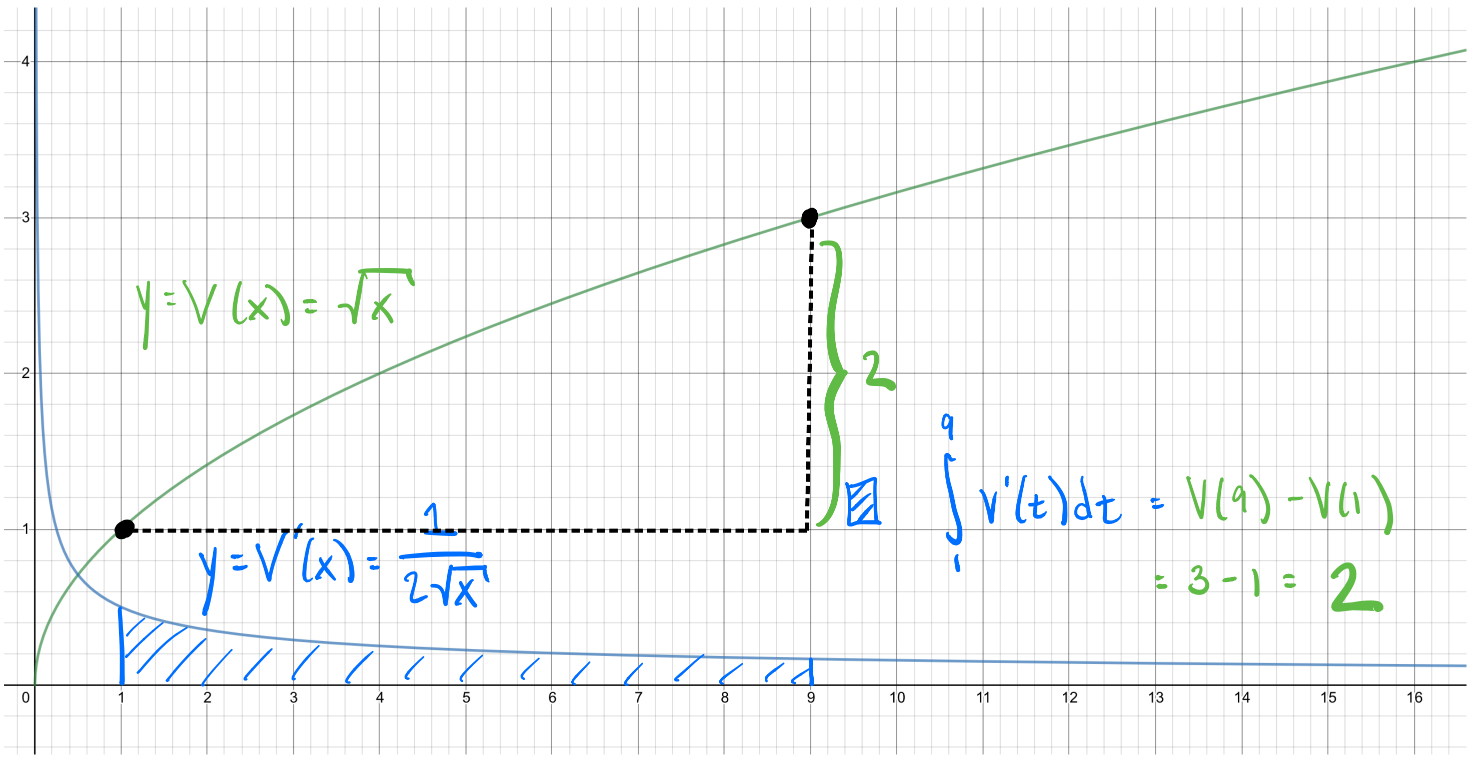
Net Change Theorem The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

- If $V(t)$ is the volume of water in a reservoir at time t, then its derivative $V'(t)$ is the rate at which water flows into the reservoir at time t. So

$$\int_{t_1}^{t_2} V'(t)\,dt = V(t_2) - V(t_1)$$

is the change in the amount of water in the reservoir between time t_1 and time t_2.



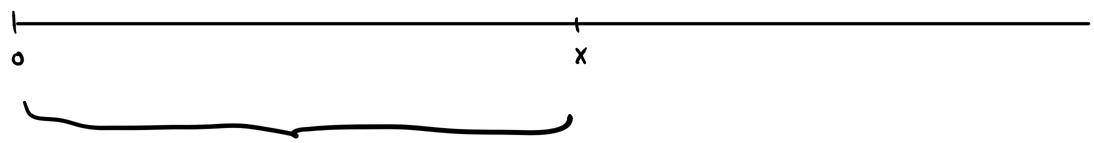
- If the mass of a rod measured from the left end to a point x is $m(x)$, then the linear density is $\rho(x) = m'(x)$. So

$$\int_a^b \rho(x)\,dx = m(b) - m(a)$$

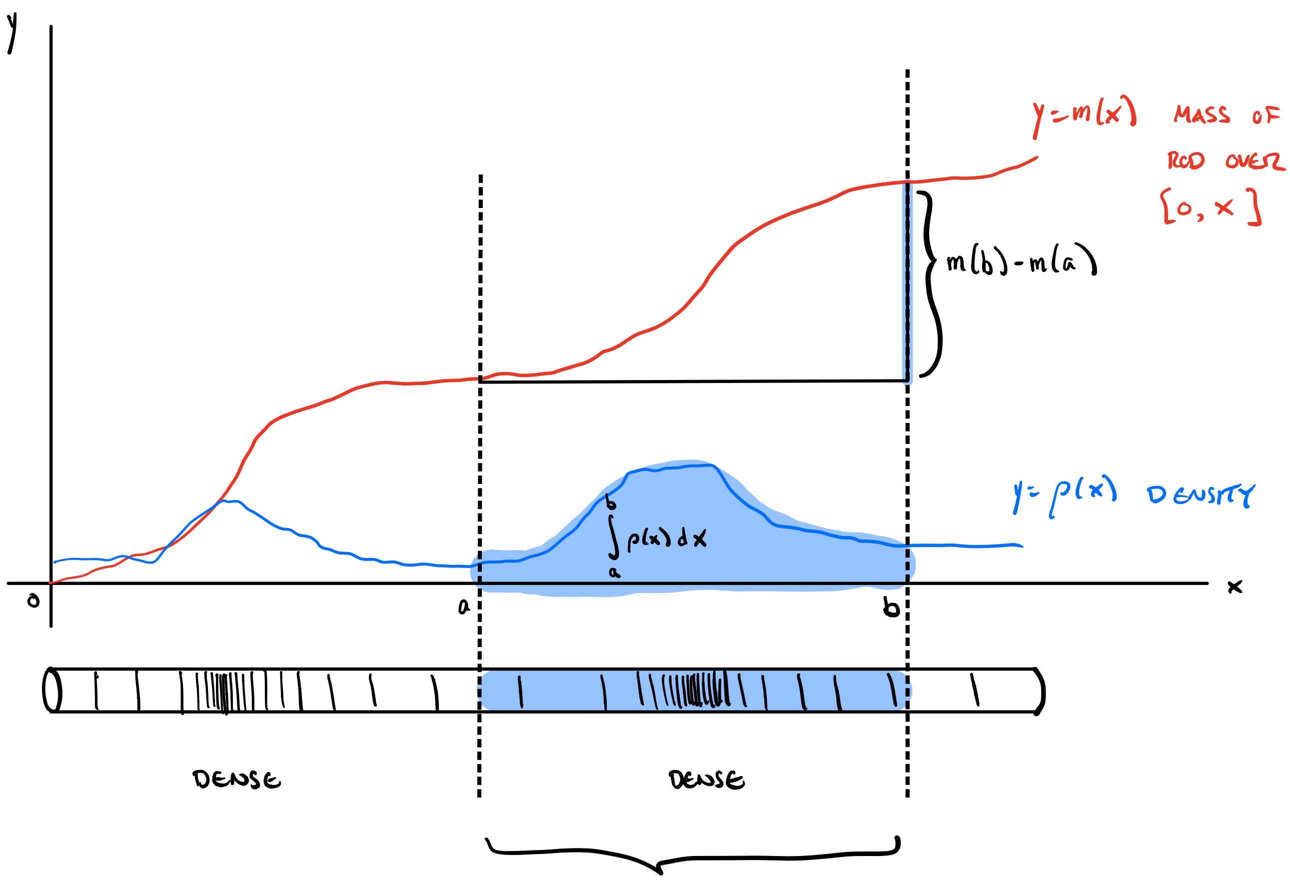
is the mass of the segment of the rod that lies between $x = a$ and $x = b$.

NOTE: HERE WE REFER TO 1D DENSITY

e.g. LBS/ft , kg/m



m(x) IS MASS OF ROD FROM 0 TO x.



MASS OF ROD BETWEEN $x = a$ & $x = b$

$$= \int_a^b \rho(x)\,dx = m(b) - m(a)$$