Use the form of the definition of the integral given in the theorem to evaluate the integral.

$$\int_1^8 (x^2 - 4x + 7)\, dx$$

280/3

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$\Delta x = \frac{b-a}{n}, \quad x_i = a + i\Delta x$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left((x_i)^2 - 4x_i + 7\right)\frac{8-1}{n}, \quad x_i = 1 + \frac{7i}{n}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(1+\frac{7i}{n}\right)^2 - 4\left(1+\frac{7i}{n}\right) + 7\right)\frac{8-1}{n}, \quad x_i = 1 + \frac{7i}{n}$$

$$\lim_{n\to\infty} \left[\sum_{i=1}^{n} \left(1 + \frac{14i}{n} + \frac{49i^2}{n^2} - 4 - \frac{28i}{n} + 7\right)\frac{7}{n}\right]$$

$$\lim_{n\to\infty} \left[\frac{7}{n} \sum_{i=1}^{n} \left(4 - \frac{14i}{n} + \frac{49i^2}{n^2}\right)\right]$$

$$\lim_{n\to\infty} \left[\frac{7}{n}\left(4\sum_{i=1}^{n} 1 - \frac{14}{n}\sum i + \frac{49}{n^2}\sum i^2\right)\right]$$

$$\lim_{n\to\infty} \left[\frac{28}{n}\sum_{i=1}^{n} 1 - \frac{7\cdot 14}{n^2}\sum_{i=1}^{n} i + \frac{7\cdot 49}{n^3}\sum_{i=1}^{n} i^2\right]$$

$$\lim_{n\to\infty}\left[\frac{28}{n}\cdot n - \frac{7^2\cdot 2}{n^2}\,\frac{n(n+1)}{2} + \frac{7^3}{n^3}\,\frac{n(n+1)(2n+1)}{6}\right]$$

See Notes

$$\lim_{n\to\infty}\left[28 - \frac{\cancel{2}\cdot 7^2}{n^2}\,\frac{n^2+n}{\cancel{2}} + \frac{7^3}{n^3}\,\frac{2n^3+3n^2+n}{6}\right]$$

$$\lim_{n\to\infty}\left[28 - 7^2\left(\frac{n^2}{n^2}+\frac{n}{n^2}\right) + \frac{7^3}{6}\left(\frac{2n^3}{n^3}+\frac{3n^2}{n^3}+\frac{n}{n^3}\right)\right]$$

$$\left(1 + 0\right) \qquad 2 \qquad 0 \qquad 0$$

$$28 - 7^2 + \frac{7^3}{6}(2)$$

$$28 - 49 + \frac{7^3}{3} = 93\tfrac{1}{3} \text{ or } \frac{280}{3}$$

§6.1 Inverse Functions

I. Precalculus

- One-to-one functions
- Horizontal line test
- Inverse functions
- Graphs

II Calculus

- Inverse Function Theorem

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1 Definition A function f is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

Different inputs produce different outputs.

IF THE OUTPUTS ARE EQUAL, THEN THE INPUTS ARE EQUAL.

3 ways to say the same thing.

ex. $f(x) = 3 + 4x$ is 1:1 but $f(x) = x^2 + 2x + 3$ is not 1:1

If $f(a) = f(b)$ (show $a = b$)

$$3 + 4a = 3 + 4b$$
$$4a = 4b$$
$$a = b \quad \checkmark$$

If $f(a) = f(b)$ (show $a = b$?)

Note: $f(x) = (x+1)^2 + 2$

Completing the sq. $\quad x^2 + 2x + 1 + 2$

$$(a+1)^2 + 2 = (b+1)^2 + 2$$
$$(a+1)^2 = (b+1)^2$$
$$a + 1 = \pm(b+1)$$
$$a = -1 \pm (b+1) = \begin{cases} b \\ -2-b \end{cases}$$

a & b can be different. f is not 1:1.

Horizontal Line Test A function is one-to-one if and only if no horizontal line intersects its graph more than once.

$\underbrace{\text{DIFFERENT INPUTS}}_{\text{DIFF } x\text{- COORD}}$ PRODUCE $\underbrace{\text{DIFFERENT OUTPUTS.}}_{\text{DIFF. } y\text{- COORD.}}$

DIFF x- COORD $\Rightarrow$ DIFF. y- COORD.

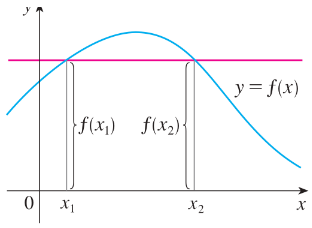


FIGURE 2
This function is not one-to-one because $f(x_1) = f(x_2)$.

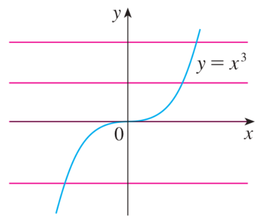


FIGURE 3
$f(x) = x^3$ is one-to-one.

2 Definition Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any y in B.

NOTATION: $f^{-1}(x)$ "f- INVERSE"

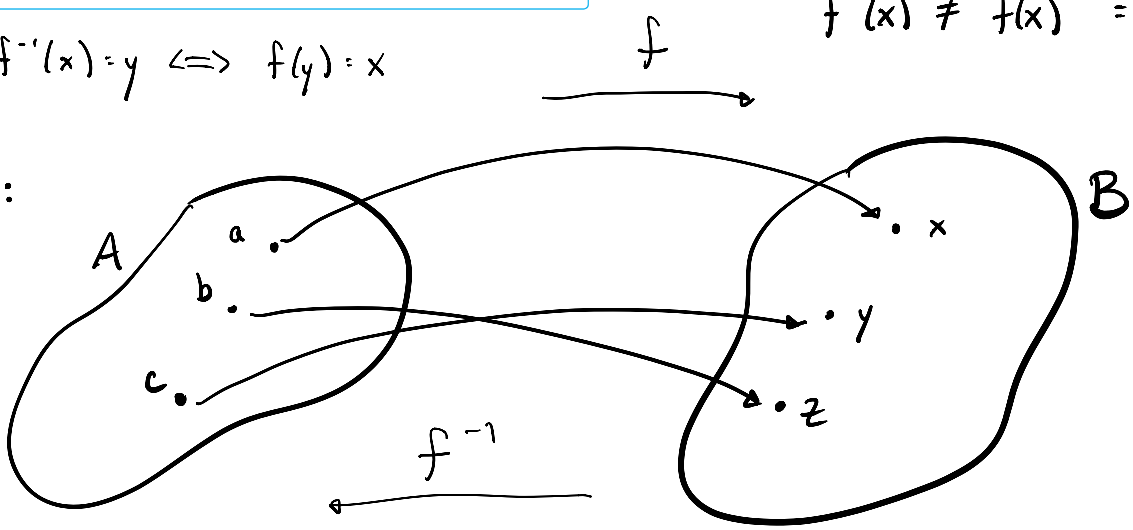
$f^{-1}(x) \neq f(x)^{-1} = \frac{1}{f(x)}$

$f^{-1}(x) = y \iff f(y) = x$

SET DIAGRAM:

$f(a) = x$

$f^{-1}(x) = a$



RUN f IN REVERSE

$f^{-1}(f(x)) = x$ for every x in A

$f(f^{-1}(x)) = x$ for every x in B

$$\begin{tikzcd} A = \mathrm{Dom}(f) \ni x \arrow[r, bend left, "f"] & f(x) \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

INVERSE FUNCTIONS COMPOSED

CANCEL

$$\begin{tikzcd} f^{-1}(x) \arrow[r, bend left, "f"] & x \in \mathrm{Dom}(f^{-1}) = B \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

17. Assume that f is a one-to-one function.
(a) If $f(6) = 17$, what is $f^{-1}(17)$?
(b) If $f^{-1}(3) = 2$, what is $f(2)$?

18. If $f(x) = x^5 + x^3 + x$, find $f^{-1}(3)$ and $f(f^{-1}(2))$.

19. If $h(x) = x + \sqrt{x}$, find $h^{-1}(6)$.

$$\begin{tikzcd} 2 \arrow[r, bend right, "f"'] & 3 \arrow[l, bend right, "f^{-1}"'] \end{tikzcd}$$

17 (a) $f(6) = 17$

$$\cancel{f^{-1}}(\cancel{f}(6)) = f^{-1}(17)$$

$$6 = f^{-1}(17)$$

(b) $f^{-1}(3) = 2$

$$f(f^{-1}(3)) = f(2)$$

$$3 = f(2)$$

5 How to Find the Inverse Function of a One-to-One Function f

STEP 1 Write $y = f(x)$.

STEP 2 Solve this equation for x in terms of y (if possible).

STEP 3 To express f^{-1} as a function of x, interchange x and y. The resulting equation is $y = f^{-1}(x)$.

23–28 Find a formula for the inverse of the function.

23. $f(x) = 5 - 4x$

24. $f(x) = \dfrac{4x - 1}{2x + 3}$

25. $f(x) = 1 + \sqrt{2 + 3x}$

26. $y = x^2 - x,\ x \geqslant \frac{1}{2}$

27. $y = \dfrac{1 - \sqrt{x}}{1 + \sqrt{x}}$

28. $f(x) = 2x^2 - 8x,\ x \geqslant 2$

25. FIND $f^{-1}(x)$ WHEN $f(x) = 1 + \sqrt{2+3x}$

1. $y = 1 + \sqrt{2+3x}$

2. $y - 1 = \sqrt{2+3x}$

$$(y-1)^2 = 2+3x$$

$$x = \frac{(y-1)^2 - 2}{3}$$

3. $y = \dfrac{(x-1)^2 - 2}{3} \Rightarrow \boxed{f^{-1}(x) = \dfrac{(x-1)^2 - 2}{3}}$

Note: $f(x) = 1 + \sqrt{2+3x}$ $\qquad f^{-1}(x) = \dfrac{(x-1)^2 - 2}{3}$

1. MULTIPLY BY 3	1. SUB 1
2. ADD 2	2. SQUARE
3. SQRT	3. SUB 2
4. ADD 1	4. DIV BY 3

TIE A KNOT $\qquad$ UNTIE A KNOT

The principle of interchanging x and y to find the inverse function also gives us the method for obtaining the graph of f^{-1} from the graph of f. Since $f(a) = b$ if and only if $f^{-1}(b) = a$, the point (a, b) is on the graph of f if and only if the point (b, a) is on the graph of f^{-1}. But we get the point (b, a) from (a, b) by reflecting about the line $y = x$. (See Figure 8.)

$f(a) = b$

(a,b) IS ON GRAPH $y = f(x)$

$f^{-1}(b) = a$

(b,a) IS ON GRAPH $y = f^{-1}(x)$

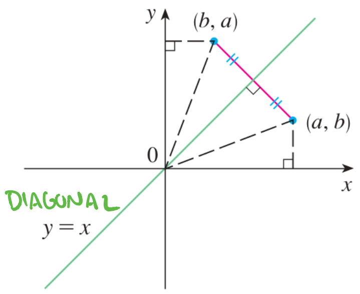


FIGURE 8

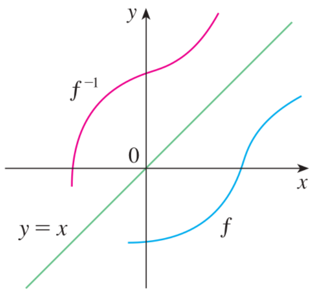


FIGURE 9

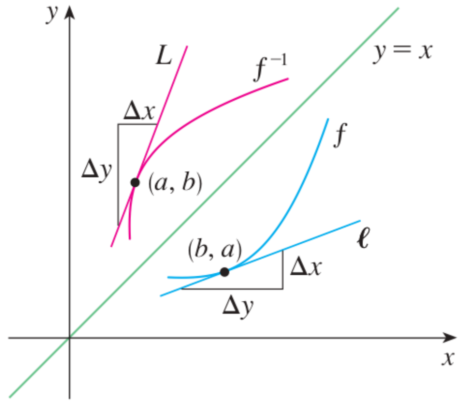
Therefore, as illustrated by Figure 9:

> The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

6 Theorem If f is a one-to-one continuous function defined on an interval, then its inverse function f^{-1} is also continuous.

7 Theorem If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

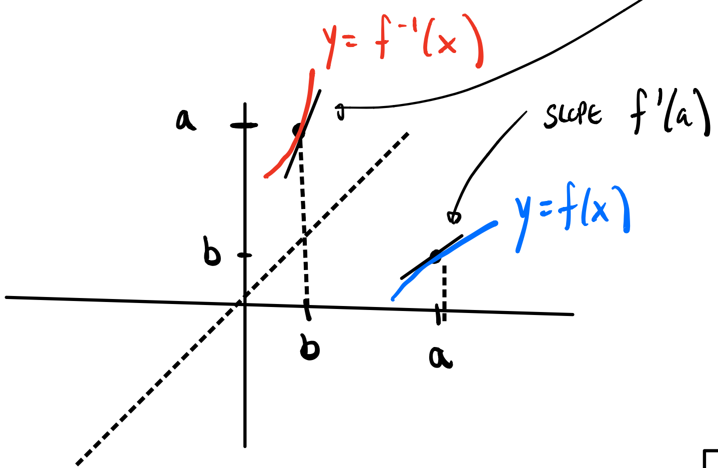


LET $f(a) = b \iff a = f^{-1}(b)$

$$(f^{-1})'(b) = \frac{1}{f'(a)}$$

f INCR $\Rightarrow$ f^{-1} INCR
f DECR $\Rightarrow$ f^{-1} DECR.

SLOPE $(f^{-1})'(b)$



$\left.\begin{matrix} \text{SLOPE } (f^{-1})'(b) \\ \text{SLOPE } f'(a) \end{matrix}\right\}$ SLOPES ARE RECIPRICALS.

$$(f^{-1})'(b) = \frac{1}{f'(a)}$$

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

I.F.T.

$$\boxed{(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}}$$

PROOF USING IMPLICIT DIFFERENTIATION:

WRITE $y = f^{-1}(x)$. FIND y'.

$\rightarrow$ $f(y) = x$ USE IMPLICIT DIFFERENTIATION

$f'(y)\, y' = 1$ CHAIN RULE

$$y' = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$$

✓

39–42 Find $(f^{-1})'(a)$.

39. $f(x) = 3x^3 + 4x^2 + 6x + 5, \quad a = 5$

40. $f(x) = x^3 + 3 \sin x + 2 \cos x, \quad a = 2$

41. $f(x) = 3 + x^2 + \tan(\pi x/2), \quad -1 < x < 1, \quad a = 3$

42. $f(x) = \sqrt{x^3 + 4x + 4}, \quad a = 3$

43. Suppose f^{-1} is the inverse function of a differentiable function f and $f(4) = 5$, $f'(4) = \frac{2}{3}$. Find $(f^{-1})'(5)$.

44. If g is an increasing function such that $g(2) = 8$ and $g'(2) = 5$, calculate $(g^{-1})'(8)$.

45. If $f(x) = \int_3^x \sqrt{1 + t^3}\, dt$, find $(f^{-1})'(0)$.

$\boxed{7}$ **Theorem** If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

39. $(f^{-1})'(5) = \dfrac{1}{f'(f^{-1}(5))}$

(1) $f'(x) = 9x^2 + 8x + 6$

(2) $f^{-1}(5) = w \iff f(w) = 5$

$3w^3 + 4w^2 + 6w + 5 = 5$

$w = 0$

$f^{-1}(5) = 0 \quad (f(0) = 5)$

$(f^{-1})'(5) = \dfrac{1}{f'(f^{-1}(5))} = \dfrac{1}{9(0)^2 + 8(0) + 6} = \dfrac{1}{6}$

43. Suppose f^{-1} is the inverse function of a differentiable function f and $f(4) = 5$, $f'(4) = \frac{2}{3}$. Find $(f^{-1})'(5)$.

44. If g is an increasing function such that $g(2) = 8$ and $g'(2) = 5$, calculate $(g^{-1})'(8)$.

45. If $f(x) = \int_3^x \sqrt{1 + t^3}\, dt$, find $(f^{-1})'(0)$.

43. $(f^{-1})'(5) = \dfrac{1}{f'(f^{-1}(5))} = \dfrac{1}{f'(4)} = \dfrac{1}{2/3} = \dfrac{3}{2}$

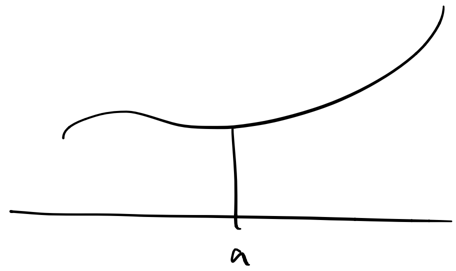
45. $(f^{-1})'(0) = \dfrac{1}{f'(f^{-1}(0))}$

$f'(x) = \sqrt{1 + x^3}$ (FTC)

$f^{-1}(0) = w = 3$

$f(w) = \int_3^w \sqrt{1+t^3}\, dt = 0$

$$\int_a^a f(x)\,dx = 0$$



$$(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'(3)} = \frac{1}{\sqrt{1+3^3}}$$

$$= \frac{1}{\sqrt{28}}$$