11/13 § 4.3 The Fundamental Theorem of Calculus. (FTC)

The FTC part 1

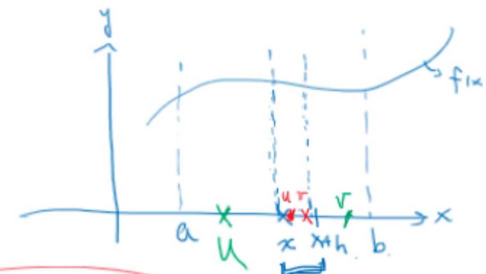
If f is continuous on $[a, b]$, then the function

$$g(x) = \int_a^x f(t)\,dt, \quad a \le x \le b$$

is continuous on $[a, b]$, and differentiable on (a, b).

and $g'(x) = f(x)$.

(idea $g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$)



PROOF: If x and $x+h$ are in (a, b)

then $g(x) = \int_a^x f(t)\,dt$.

$$g(x+h) = \int_a^{x+h} f(t)\,dt$$

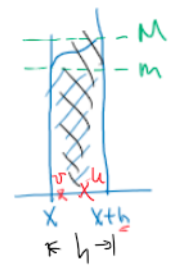
$$g(x+h) - g(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt$$

$$= \int_x^{x+h} f(t)\,dt \quad \text{for } h \neq 0$$

— From the closed interval method, we know that there are abs. max. $f(u)$ and abs. min. $f(v)$ where u & v are both in $[x, x+h]$

let $\underline{m} = f(v)$, $\underline{M} = f(u)$

$$\Rightarrow \quad m \leq f(x) \leq M \quad \text{for } x \text{ in } [x, x+h]$$



integrate all terms →

$$\Rightarrow \quad \int_x^{x+h} m\,dt \leq \int_x^{x+h} f(t)\,dt \leq \int_x^{x+h} M\,dt$$

$$\Rightarrow \quad mh \leq \int_x^{x+h} f(t)\,dt \leq Mh$$

$\div h$

$$\Rightarrow \quad m \leq \frac{1}{h}\int_x^{x+h} f(t)\,dt \leq M$$

$$\Rightarrow \quad \overset{f(v)}{\overset{\shortparallel}{m}} \leq \frac{g(x+h) - g(x)}{h} \leq M \leftarrow f(u)$$

$\lim_{h \to 0}$

$$\Rightarrow \quad \lim_{h\to 0} f(v) \leq \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \leq \lim_{h\to 0} f(u)$$

as $h \to 0$
$u \to x$
$v \to x$
$x+h \to x$

$$\Rightarrow \quad f(x) \leq g'(x) \leq f(x)$$

Therefore, $g'(x) = f(x)$ ∎

$f(x) = \sqrt{1+x^2}$

Example 1. ① If $g(x) = \int_0^x \sqrt{1+t^2}\,dt$. find $g'(x)$

$\Rightarrow \quad g'(x) = f(x) = \sqrt{1+x^2}$

② If $S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right)dt$. find $S'(x)$

$\Rightarrow \quad S'(x) = \sin\left(\frac{\pi x^2}{2}\right)$.

Example 2. Find $\frac{d}{dx}\int_1^{x^4} \sec t\,dt$

think about composite function

$g(x) = \int_1^{x^4} \sec t\,dt$

let $u = x^4$

$\frac{du}{dx} = 4x^3$

$g(u(x)) = \int_1^{u} \sec t\,dt$

FTC

$\Rightarrow \frac{d}{dx}\left(g(u)\right) = \frac{d}{du}\left(\int_1^u \sec t\,dt\right)\frac{du}{dx} = \sec u\,(4x^3)$

$= 4x^3 \sec(x^4)$.

The FTC part 2

$$\int_a^b f(x)\,dx = F(b)-F(a)$$

where $F(x)$ is any antiderivative of $f(x)$ or $F'(x)=f(x)$

PROOF: Let $g(x)=\int_a^x f(t)\,dt$, then $g'(x) = f(x)$.

is an $g(x)$ is an antiderivative of $f(x)$.

this also implies that $g(x) = F(x) + C$

if $x=a$: $g(a)=\int_a^a f(t)\,dt = 0$, if $x=b$: $g(b) = \int_a^b f(t)\,dt$

$\Rightarrow F(a) + C = 0$. $\Rightarrow F(b) + C = \int_a^b f(t)\,dt$

$\Rightarrow C = -F(a)$

Finally, we have $\int_a^b f(x)\,dx = F(b)-F(a)$

Remark: we usually write $\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b)-F(a)$.

Example 3. ① $\int_{-2}^{1} x^3\,dx = \frac{1}{4}x^4 \Big|_{-2}^{1}$ $\quad f(x) = 1\cdot x^3$

$F(x) = \frac{1}{4}x^4$

$= \frac{1}{4}(1)^4 - \frac{1}{4}(-2)^4 = \frac{1}{4} - \frac{1}{\cancel{4}}\cdot \overset{4}{\cancel{16}} = \frac{-15}{4}$ #

② Find the area under $y = \cos x$ from 0 to b.

where $0 \le b \le \frac{\pi}{2}$.

$\int_0^b \cos x\,dx = \sin x \Big|_0^b = \sin(b) - \cancel{\sin 0}^{\,0}$

$= \sin(b)$ #