Final Exam, 19 December 2016
Math 201

Name: * ANSWER KEY * Instructor: ADAMSKI

[1]10	[2]10	[3]10	[4]10	[5]10	[6]7	[7]10	[8]10	[9]10	[10]13	TOTAL

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Instructions: Please read each question carefully, show all work, and check afterwards that you have answered all of each question correctly. **Important: No books, calculators, or notes are allowed. Turn off cell phones, alarms, and anything else that makes noises!** You must show **all** your work to receive credit. Any crossed out work will be disregarded (even if correct). Write **one** clear answer with a coherent derivation for each question. Good luck!

[1] (2.5 pts each) Compute $f'(x)$ for each of the functions below:

(a) $f(x) = \cos(x + \frac{1}{x})$

CHAIN RULE:

$$f'(x) = \sin(x + \tfrac{1}{x}) \cdot \frac{d}{dx}\left[x + \frac{1}{x}\right]$$

$$\boxed{f'(x) = \sin(x+\tfrac{1}{x})(1 - \tfrac{1}{x^2})}$$

(b) $f(x) = (x^2 + 3\sin x)(3\sqrt{x} - \frac{1}{x^2})$

PRODUCT RULE:

$$f'(x) = (3\sqrt{x} - \tfrac{1}{x^2}) \cdot \frac{d}{dx}\left[x^2 + 3\sin x\right] + (x^2 + 3\sin x) \cdot \frac{d}{dx}\left[3\sqrt{x} - \frac{1}{x^2}\right]$$

$$\boxed{f'(x) = (3\sqrt{x} - \tfrac{1}{x^2})(2x + 3\cos x) + (x^2 + 3\sin x)(\tfrac{3}{2\sqrt{x}} + \tfrac{2}{x^3})}$$

(c) $f(x) = \frac{x^2 - 5x^5 + x^7}{x^6}$

QUOTIENT RULE:

$$f'(x) = \frac{(x^6) \cdot \frac{d}{dx}\left[x^2 - 5x^5 + x^7\right] - (x^2 - 5x^5 + x^7) \cdot \frac{d}{dx}\left[x^6\right]}{(x^6)^2}$$

$$\boxed{f'(x) = \frac{x^6(2x - 25x^4 + 7x^6) - (x^2 - 5x^5 + x^7)(6x^5)}{x^{12}}}$$

(d) $f(x) = \frac{1+\sec x}{\tan x}$

QUOTIENT RULE:

$$f'(x) = \frac{(\tan x) \cdot \frac{d}{dx}\left[1 + \sec x\right] - (1 + \sec x) \cdot \frac{d}{dx}\left[\tan x\right]}{(\tan x)^2}$$

$$\boxed{f'(x) = \frac{(\tan x)(\sec x \tan x) - (1 + \sec x)(\sec^2 x)}{\tan^2 x}}$$

OR $$\frac{\tan^2 x \sec x - \sec^2 x - \sec^3 x}{\tan^2 x}$$

OR $\sec x - \csc^2 x - \sec x \csc^2 x$

$= \sec x (1 - \csc^2 x) - \csc^2 x$

$= -\sec x \cot^2 x - \csc^2 x$

$\vdots$

∞ MANY WAYS TO WRITE THIS ü

[1] (10 pts)

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[2] **(2.5 pts each)** Find each integral:

(a) $\int (4x-5)\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx$

EXPAND : $\int 4x^{3/2} + 4x^{1/2} - 5x^{1/2} - 5x^{-1/2} \, dx$

$= \int 4x^{3/2} - x^{1/2} - 5x^{-1/2} \, dx$

POWER RULE : $4 \cdot \frac{1}{\frac{3}{2}+1} x^{\frac{3}{2}+1} - \frac{1}{\frac{1}{2}+1} x^{\frac{1}{2}+1} - 5 \cdot \frac{1}{-\frac{1}{2}+1} x^{-\frac{1}{2}+1} + C$

$= \boxed{\frac{8}{5} x^{5/2} - \frac{2}{3} x^{3/2} - 10 x^{1/2} + C}$

(b) $\int_0^{\pi/2} (\cos^3(3x) - \cos(3x))dx$

FACTOR : $\int_0^{\pi/2} (\cos^2(3x) - 1)\cos(3x) \, dx$

TRIG Id : $\int_0^{\pi/2} -\sin^2(3x)\cos(3x) \, dx$

LET $u = \sin(3x)$

$du = 3\cos(3x)\,dx$

$-\frac{1}{3} du = -\cos(3x)\,dx$

AND $\int_{x=0}^{x=\pi/2} \rightsquigarrow \int_{u=0}^{u=-1}$

$\rightsquigarrow -\frac{1}{3}\int_0^{-1} u^2 \, du = -\frac{1}{3}\cdot\frac{1}{3}u^3 \Big|_0^{-1} = -\frac{1}{9}\left[(-1)^3 - (0)^3\right]$

$= \boxed{\frac{1}{9}}$

(c) $\int \frac{x^4}{\sqrt{2x^5+127}}dx$

LET $u = 2x^5 + 127$

$du = 10x^4\,dx$

$\frac{1}{10}\,du = x^4\,dx$

$\rightsquigarrow \frac{1}{10}\int u^{-1/2}\,du = \frac{1}{10}\cdot 2u^{1/2} + C$

$\rightsquigarrow \boxed{\frac{1}{5}\sqrt{2x^5 + 127} + C}$

(d) $\int \frac{\left(1+\frac{2}{x^2}\right)^5}{x^3}dx$

LET $u = 1 + \frac{2}{x^2}$

$du = -\frac{4}{x^3}\,dx$

$-\frac{1}{4}\,du = \frac{1}{x^3}\,dx$

$\rightsquigarrow -\frac{1}{4}\int u^5\,du$

$= -\frac{1}{4}\cdot\frac{1}{6}u^6 + C$

$\rightsquigarrow \boxed{-\frac{1}{24}\left(1+\frac{2}{x^2}\right)^6 + C}$

[2] (10 pts)

Please leave blank!

[3] **(2.5 pts each)** Find the limits, or state that the limit does not exist (you must justify your answer):

(a) $\lim_{x\to 3} \frac{x^2-3x}{x^2+x-12}$

$$= \lim_{x\to 3} \frac{x\cancel{(x-3)}}{(x+4)\cancel{(x-3)}}$$

$$= \lim_{x\to 3} \frac{x}{x+4} = \frac{3}{3+4} = \boxed{\frac{3}{7}}$$

(b) $\lim_{x\to +\infty} \frac{2x^3+3x}{5x^3-x^2+27}$

$$= \lim_{x\to\infty} \frac{\cancel{x^3}\left(2+\frac{3}{x^2}\right)}{\cancel{x^3}\left(5-\frac{1}{x}+\frac{27}{x^3}\right)}$$

$$= \frac{\lim_{x\to\infty}\left(2+\frac{3}{x^2}\right)}{\lim_{x\to\infty}\left(5-\frac{1}{x}+\frac{27}{x^3}\right)} = \frac{2+0}{5-0+0}$$

$$= \boxed{\frac{2}{5}}$$

(c) $\lim_{x\to+\infty} \frac{1}{x^2}\sin(x)$

SQUEEZE THM:

SINCE $-1 \leq \sin x \leq 1$

$\Rightarrow -\frac{1}{x^2} \leq \frac{1}{x^2}\sin x \leq \frac{1}{x^2}$

WE HAVE $\underbrace{\lim_{x\to\infty} \frac{-1}{x^2}}_{0} \leq \lim_{x\to\infty} \frac{1}{x^2}\sin x \leq \underbrace{\lim_{x\to\infty} \frac{1}{x^2}}_{0}$.

THUS, BY SQUEEZE THM, $\lim_{x\to\infty} \frac{1}{x^2}\sin x = \boxed{0}$.

(d) $\lim_{x\to 0} \frac{\tan x}{x+\sin x}$

$= \lim_{x\to 0} \frac{1}{\cos x} \cdot \frac{\sin x}{x+\sin x} = \underbrace{\lim_{x\to 0} \frac{1}{\cos x}}_{1} \cdot \lim_{x\to 0} \frac{\sin x}{x+\sin x}$

$= \lim_{x\to 0} \dfrac{\frac{\sin x}{x}}{1+\frac{\sin x}{x}} = \dfrac{\lim_{x\to 0}\frac{\sin x}{x}}{1+\lim_{x\to 0}\frac{\sin x}{x}}$

$= \frac{1}{1+1} = \boxed{\frac{1}{2}}$

[3] (10 pts)

Please leave blank!

[4] (10 pts)

(a) (5 pts) Using the limit definition of the derivative, compute $f'(x)$ if $f(x) = x^2 - 2x$ (no credit will be given for any other method).

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - 2(x+h) - (x^2 - 2x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{2x} - 2h - \cancel{x^2} + \cancel{2x}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x + h - 2)}{\cancel{h}} = \boxed{2x - 2}$$

(c) (5 pts) Find an equation of the tangent line to the graph $y = x^2 - 2x$ at the point $(1, -1)$.

POINT - SLOPE FORMULA : $y - (-1) = m(x - 1)$

$m = y'$ AT $x = 1$

$y' = 2x - 2$ SO $m = 2(1) - 2 = 0$

$\therefore y - (-1) = 0$

OR $\boxed{y = -1}$

[5] (10 pts)

Please leave blank!

[5] (10 pts)

(a) (5 pts) Let $F(x) = \int_{x^2}^{\sin(x)} \sqrt{1-t^3}\,dt$. Find $F'(x)$.

$$F(x) = \int_{x^2}^{0} \sqrt{1-t^3}\,dt + \int_{0}^{\sin x} \sqrt{1-t^3}\,dt$$

$$= -\int_{0}^{x^2} \sqrt{1-t^3}\,dt + \int_{0}^{\sin x} \sqrt{1-t^3}\,dt$$

BY F.T.C., $\boxed{F'(x) = -2x\sqrt{1-x^6} + \cos x\sqrt{1-\sin^3 x}}$

IMPLICIT EQ

(b) (5 pts) Find the equation of the tangent line to the curve $x^2 + 2xy - y^2 + x = 2$ at the point $(1, 2)$. # IMPLICIT DIFFERENTIATION

$$\frac{d}{dx}[x^2] + \underbrace{\frac{d}{dx}[2x]\cdot y + 2x\cdot\overbrace{\frac{d}{dy}[y]\cdot\frac{dy}{dx}}}_{\text{PRODUCT RULE}} - \frac{d}{dy}[y^2]\cdot\frac{dy}{dx} + \frac{d}{dx}[x] = \frac{d}{dx}[2]$$

CHAIN RULE

$$\Rightarrow 2x + 2y + 2x\frac{dy}{dx} - 2y\frac{dy}{dx} + 1 = 0$$

$$\frac{dy}{dx} = \frac{-1-2x-2y}{2x-2y} \xrightarrow[(1,2)]{\text{AT}} = \frac{-1-2(1)-2(2)}{2(1)-2(2)}$$

$$= \frac{-7}{-2} = \frac{7}{2}$$

$\therefore$ $\boxed{y - 2 = \frac{7}{2}(x-1)}$ or $y = \frac{7}{2}x - \frac{3}{2}$

[5] (10 pts)

Please leave blank!

[6] **(7 pts)** When a circular plate is heated in an oven, its radius increases at the rate of 0.01 cm/min. At what rate is the plate's area increasing when the radius is 50cm? Be sure to include units in your answer.

RELATED RATES

AREA $A = \pi r^2$

$\downarrow$ TAKE DERIVATIVES WITH RESPECT TO TIME, t

$$\frac{dA}{dt} = \pi \cdot \frac{d}{dr}[r^2] \cdot \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$r = 50$ cm

$\frac{dr}{dt} = 0.01$ cm/MIN

$$\frac{dA}{dt} = 2\pi(50)(.01) = 2\pi(.5) = \pi$$

$$\frac{dA}{dt} = \boxed{\pi \text{ cm}^2/\text{MIN}}$$

[6] (7 pts)

Please leave blank!

[7] (5 pts each)

LINEAR APPROXIMATION

(a) Find an approximation for tan(0.01) using calculus.

SINCE $f'(a) \approx \dfrac{f(x) - f(a)}{x - a}$ WHEN x IS CLOSE TO a,

SETTING $f(x) = \tan x$, $a = 0$, AND $x = 0.01$,

WE HAVE $\tan(0.01) \approx \sec^2(0)(0.01) + \tan(0)$

$\approx \boxed{0.01}$

(b) Suppose $f(x)$ is a differentiable function such that $f(1) = 2$ and $f'(x) \le 5$ for all x. What is the largest possible value for $f(4)$?

BY THE MEAN VALUE THEOREM, THERE EXISTS A NUMBER c, $1 < c < 4$,

SUCH THAT $f'(c) = \dfrac{f(4) - f(1)}{4 - 1} = \dfrac{f(4) - 2}{3}$.

THAT IS, $f(4) = 3f'(c) + 2 \le 3(5) + 2 = 17$

$\boxed{f(4) \le 17}$

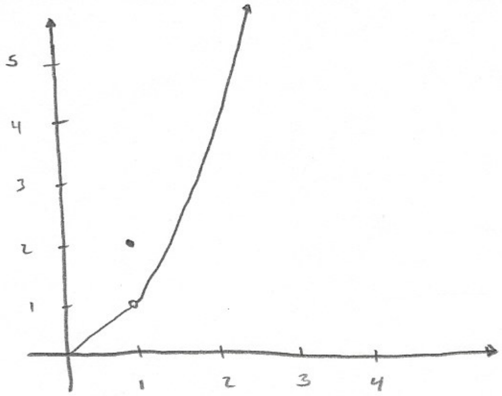
[7] (20 pts)

Please leave blank!

[8] (10 pts)

(a) (2 pts each) Let $f(x) = \begin{cases} x & \text{if } x < 1, \\ 2 & \text{if } x = 1, \\ x^2 & \text{if } x > 1. \end{cases}$

(i) Sketch the graph of $y = f(x)$ for $x \in [0, 4]$.



(ii) Find $\lim_{x \to 1} f(x)$.

$$\left.\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} x = 1 \\ \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} x^2 = 1 \end{aligned}\right\} \therefore \lim_{x \to 1} f(x) = \boxed{1}$$

(iii) Is $f(x)$ continuous at the point $x = 1$? Please justify your answer.

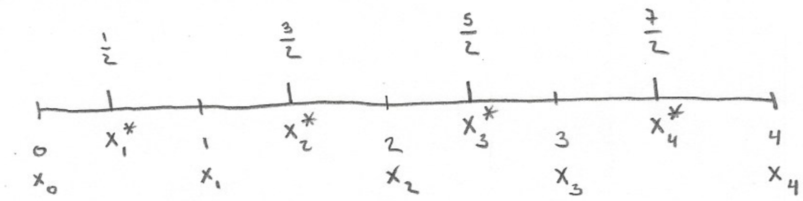
f IS CONTINUOUS AT $x = 1$ IF AND ONLY IF

$$\underbrace{\lim_{x \to 1} f(x)}_{1} = \underbrace{f(1)}_{2}$$

$1 = 2$ FALSE!

$\therefore$ f IS NOT CONTINUOUS AT $x = 1$. $\boxed{\text{No.}}$

(b) **(4 pts)** For the function f above, use a Riemann Sum to estimate $\int_0^4 f(x)dx$ by using the Midpoint Rule with 4 subdivisions.



$$M_4 = \sum_{i=1}^{4} f(x_i^*)\,\Delta x \quad , \quad x_i^* = \frac{x_{i-1} + x_i}{2} \quad (\text{MIDPOINT})$$

$$\Delta x = \frac{b-a}{n} = \frac{4-0}{4} = 1$$

$$M_4 = \underbrace{f\left(\tfrac{1}{2}\right)}_{f(x)=x} + \underbrace{f\left(\tfrac{3}{2}\right) + f\left(\tfrac{5}{2}\right) + f\left(\tfrac{7}{2}\right)}_{f(x)=x^2}$$

$$M_4 = \frac{1}{2} + \left(\frac{3}{2}\right)^2 + \left(\frac{5}{2}\right)^2 + \left(\frac{7}{2}\right)^2$$

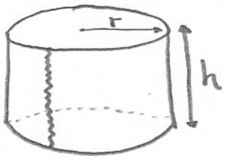
$$= \frac{2 + 9 + 25 + 49}{4} = \boxed{\frac{85}{4}}$$

[8] (10 pts)

Please leave blank!

[9] **(10 pts)** A cylindrical can is made of two different materials: the sides are made of a material that costs 1 dollar per square foot, while the top and bottom are made of a material that costs 2 dollars per square foot.

If the total volume of the can must be 1 cubic feet, find the dimensions of the can that minimize cost.

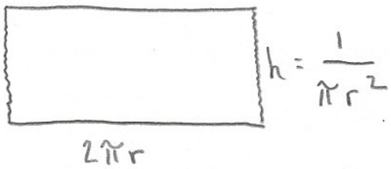


$$V = \pi r^2 h$$

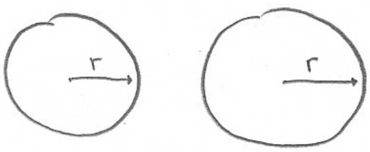
$$1 = \pi r^2 h \longrightarrow h = \frac{1}{\pi r^2}$$

CUT OPEN

SIDE :



TOPS :



SURFACE AREA $= 2\cancel{\pi}\cancel{r}\left(\frac{1}{\cancel{\pi} r \cancel{r}}\right) + 2\pi r^2$

COST : $C(r) = 1 \cdot \frac{2}{r} + 2 \cdot 2\pi r^2$

$$C(r) = \frac{2}{r} + 4\pi r^2$$

MINIMIZE THIS , $0 < r < \infty$

CRITICAL PNT(S) : $C'(r) = -\frac{2}{r^2} + 8\pi r = 0 \longrightarrow 8\pi r = \frac{2}{r^2}$

$$\longrightarrow r^3 = \frac{1}{4\pi} \longrightarrow r = \sqrt[3]{\frac{1}{4\pi}}$$

NOTE : $C''(r) = \frac{4}{r^3} + 8\pi > 0$ FOR $r > 0$,

SO $r = \sqrt[3]{\frac{1}{4\pi}}$ IS A MINIMUM ✓

[9] (10 pts)

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DIMENSIONS: $\boxed{r = \sqrt[3]{\frac{1}{4\pi}} \quad , \quad h = \frac{1}{\pi \sqrt[3]{\frac{1}{4\pi}}^{\,2}}}$

[10] **(13 pts)** For the function $f(x) = \frac{x^2}{x^2-4}$, you are given (do not compute!) that

$$f'(x) = \frac{-8x}{(x^2-4)^2} \text{ and } f''(x) = \frac{8(3x^2+4)}{(x^2-4)^3}.$$

(a) Find the domain of $f(x)$.

(b) Find the coordinates of all intercepts, and the equations of all asymptotes, of the graph of $y = f(x)$.

(c) In what intervals is the function f increasing? decreasing?

(d) In what intervals is the graph of f concave up? concave down?

(e) Find the coordinates of all local maxima, local minima, and points of inflection of the function f.

(f) Sketch the graph of $y = f(x)$. Label the features you found in items b and e.

(a) $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

(b) ONLY INTERCEPT = $(0,0)$

VERTICAL ASYMPTOTES : $x = \pm 2$

HORIZONTAL ASYMPTOTE : $y = 0$

(c) DECREASING : $(0, 2) \cup (2, \infty)$

INCREASING : $(-\infty, -2) \cup (-2, 0)$

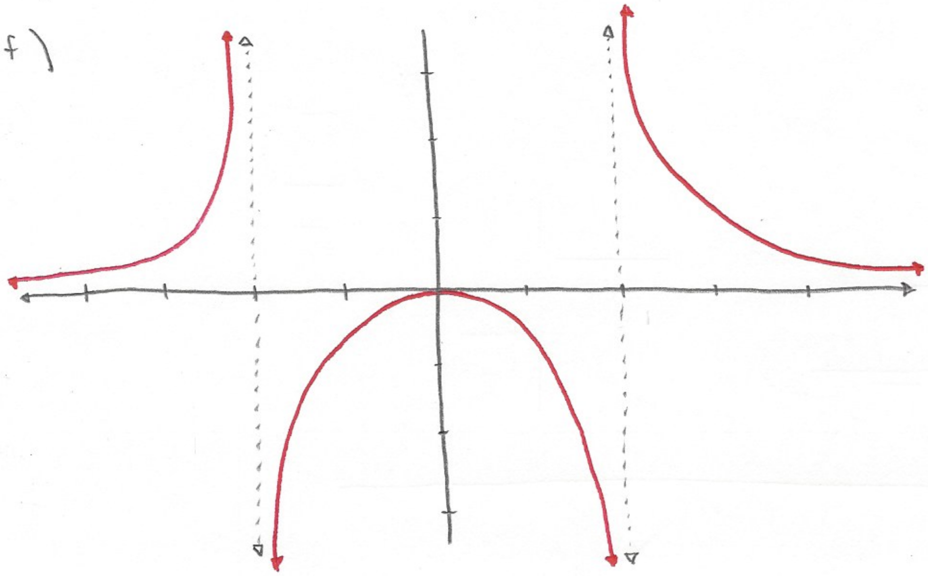
(d) CONCAVE UP : $(-\infty, -2) \cup (2, \infty)$

CONCAVE DOWN : $(-2, 2)$

(e) LOCAL MAXIMUM : $(0, 0)$

P.O.I. : NONE

(f)



[10] (13 pts)

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