Name: *ANSWER KEY* **4/2/2015**

Math 202 Midterm Exam

Directions Answer all questions in the space provided and box your final answers. Good luck!

1. (8 points) Let f be defined to be the one-to-one function

$$f(x) = \sqrt{x^3 + x^2 + x + 1}.$$

Use the inverse function theorem to find $(f^{-1})'(2)$.
Hint: You can easily find $f^{-1}(2)$ by just guessing and checking.

$$(f^{-1})'(2) = \frac{1}{f'(\underbrace{f^{-1}(2)}_{f(1)=2})} = \frac{1}{f'(1)} \quad \text{(INV. FUNC. THM.)}$$

SINCE $f'(x) = \dfrac{3x^2 + 2x + 1}{2\sqrt{x^3 + x^2 + x + 1}}$ & $f'(1) = \dfrac{3 + 2 + 1}{2\sqrt{1+1+1+1}} = \dfrac{3}{2}$

WE HAVE $(f^{-1})'(2) = \dfrac{2}{3}$

2. (8 points) Suppose a sample of radioactive material decays exponentially, and suppose it takes 6 days for the sample to decay to 88% of its initial mass. Find the half-life of the material.

$$M(t) = M_0 e^{kt}$$

$$M(6) = M_0 e^{6k} = .88 M_0$$

$$6k = \ln .88$$

$$k = (\ln .88)\frac{1}{6}$$

$$M(t) = M_0 \left(e^{\ln .88}\right)^{\frac{t}{6}} = M_0 (.88)^{t/6}$$

SOLVE FOR t: $M_0 (.88)^{t/6} = .5 M_0$

$$\frac{t}{6} \ln .88 = \ln .5$$

$$t = \frac{6 \ln .5}{\ln .88}$$

3. Differentiate each of the following functions.

(a) (8 points) $f(x) = \tan^{-1}(\sinh x)$

$$f'(x) = \frac{1}{\sinh^2 x + 1} \cdot \frac{d}{dx}\left[\sinh x\right] = \frac{\cosh x}{\sinh^2 x + 1}$$

$$= \frac{\cosh x}{\cosh^2 x} = \operatorname{sech} x$$

(b) (8 points) $g(x) = \dfrac{2^x}{x^2\sqrt{x^4+1}}$ *Hint: Use logarithmic differentiation.*

$$\ln g(x) = x\ln 2 - 2\ln x - \frac{1}{2}\ln(x^4+1)$$

$$\frac{1}{g(x)} g'(x) = \ln 2 - \frac{2}{x} - \frac{4x^3}{2(x^4+1)}$$

$$g'(x) = \frac{2^x}{x^2\sqrt{x^4+1}}\left(\ln 2 - \frac{2}{x} - \frac{2x^3}{x^4+1}\right)$$

4. (8 points) Calculate the following limit.

$$\lim_{x\to 0^+} \ln x \, \sin x \quad : \quad \infty \cdot 0 \quad \text{IND. FORM}$$

$$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sin x}} \quad : \quad \frac{\infty}{\infty} \quad \text{IND FORM}$$

$$\xrightarrow{\text{L'HÔ}} \lim_{x\to 0^+} \frac{1/x}{\frac{-\cos x}{\sin^2 x}} = \lim_{x\to 0^+} \frac{-\sin^2 x}{x\cos x} \quad : \quad \frac{0}{0} \quad \text{IND. FORM}$$

$$\xrightarrow{\text{L'HÔ}} \lim_{x\to 0^+} \frac{-2\sin x\cos x}{\cos x - x\sin x} = 0$$

5. Evaluate each of the following integrals.

(a) (8 points) $\int_1^4 \sqrt{t}\,\ln t\,dt$

$u = \ln t \qquad v = \frac{2}{3}t^{3/2}$

$du = \frac{1}{t}\,dt \qquad dv = \sqrt{t}\,dt$

$$= \frac{2}{3}t^{3/2}\ln t\Big|_1^4 - \int_1^4 \frac{2}{3}\sqrt{t}\,dt$$

$$= \frac{2}{3}(8\ln 4 - 0) - \frac{4}{9}t^{3/2}\Big|_1^4 = \frac{2}{3}(8\ln 4) - \frac{4}{9}(8-1)$$

$$= \frac{48\ln 4 - 28}{9}$$

(b) (8 points) $\int \sin^5 x \cos^3 x\,dx$ $= \int \sin^5 x\,(1-\sin^2 x)\cos x\,dx$

Let $u = \sin x$

$du = \cos x\,dx$

$$\rightarrow \int u^5(1-u^2)\,du = \int u^5 - u^7\,du = \frac{1}{6}u^6 - \frac{1}{8}u^8 + C$$

$$\rightarrow \frac{1}{6}\sin^6 x - \frac{1}{8}\sin^8 x + C$$

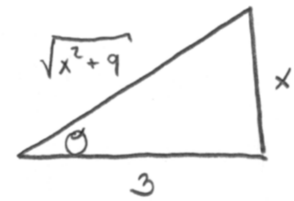
OR $-\frac{1}{8}\cos^8 x + \frac{1}{3}\cos^6 x - \frac{1}{4}\cos^4 x + C$

6. Evaluate each of the following integrals.

(a) (8 points) $\int \frac{1}{x^2\sqrt{x^2+9}}\,dx$

LET $x = 3\tan\theta$

$dx = 3\sec^2\theta\, d\theta$



$$\rightsquigarrow \int \frac{1}{(3\tan\theta)^2\sqrt{9\tan^2\theta+9}}\,3\sec^2\theta\,d\theta$$

$$= \int \frac{\cancel{3}\sec^{\cancel{2}}\theta}{9\tan^2\theta\cdot \cancel{3}\cancel{\sec\theta}}\,d\theta = \frac{1}{9}\int \frac{\cos^{\cancel{2}}\theta}{\sin^2\theta}\cdot\frac{1}{\cancel{\cos\theta}}\,d\theta$$

$$= -\frac{1}{9}\csc\theta + C = \frac{-\sqrt{x^2+9}}{9x} + C$$

(b) (8 points) $\int \frac{10}{(x-1)(x^2+9)}\,dx = \int \frac{A}{x-1} + \frac{Bx+C}{x^2+9}\,dx$

$10 = A(x^2+9) + (Bx+C)(x-1)$

$x=1$: $10 = 10A \Rightarrow A = 1$

NOW $10 = x^2+9 + (Bx+C)(x-1)$

$x=0$: $10 = 9 - C \longrightarrow C = -1$

NOW $10 = x^2+9 + (Bx-1)(x-1)$

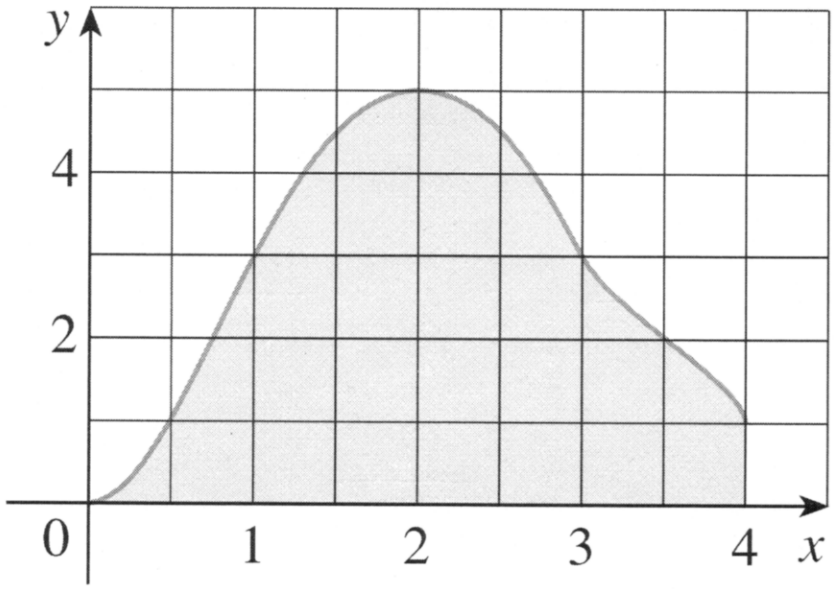
$x=2$: $10 = 4 + 9 + 2B - 1$

$-2 = 2B \longrightarrow B = -1$

$$\Rightarrow \int \frac{1}{x-1} - \frac{x}{x^2+9} - \frac{1}{x^2+9}\,dx$$

$$= \ln|x-1| - \frac{1}{2}\ln|x^2+9| - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

7. Approximate the area under the graph by using



(a) (4 points) the trapezoid rule with $n = 4$ (i.e. T_4), and

$$T_4 = \frac{\Delta x}{2} \sum_{i=1}^{4} \left(f(x_{i-1}) + f(x_i) \right), \quad \Delta x = \frac{b-a}{n} = \frac{4-0}{4} = 1$$

$$T_4 = \frac{1}{2} \left(f(0) + 2f(1) + 2f(2) + 2f(3) + f(4) \right)$$

$$T_4 = \frac{1}{2} \left(0 + 2(3) + 2(5) + 2(3) + 1 \right) = \frac{1}{2}(23) = \frac{23}{2} \text{ or } 11\tfrac{1}{2}$$

(b) (4 points) Simpson's rule with $n = 4$ (i.e. S_4).

$$S_4 = \frac{\Delta x}{3} \sum_{i=1}^{3} \left(f(x_{i-1}) + 4f(x_i) + f(x_{i+1}) \right)$$

$$S_4 = \frac{1}{3} \left(f(0) + 4f(1) + 2f(2) + 4f(3) + f(4) \right)$$

$$S_4 = \frac{1}{3} \left(0 + 4(3) + 2(5) + 4(3) + 1 \right) = \frac{1}{3}(35) = \frac{35}{3} \text{ or } 11\tfrac{2}{3}$$

8. (8 points) Evaluate the following improper integral.

$$\int_1^\infty \frac{\ln x}{x^3}\,dx$$

Hint: First use integration by parts to evaluate the indefinite integral.

$$= \lim_{t\to\infty} \int_1^t \frac{\ln x}{x^3}\,dx$$

LET $u = \ln x$ $\quad v = -\frac{1}{2x^2}$

$du = \frac{1}{x}\,dx$ $\quad dv = \frac{1}{x^3}\,dx$

$$= \lim_{t\to\infty} \left[-\frac{\ln x}{2x^2}\Big|_1^t + \frac{1}{2}\int_1^t \frac{1}{x^3}\,dx \right]$$

$$= \lim_{t\to\infty} \frac{-\ln t}{2t^2} - \frac{1}{4x^2}\Big|_1^t = \lim_{t\to\infty} \frac{-\ln t}{2t^2} - \frac{1}{4t^2} + \frac{1}{4}$$

$$= -\frac{1}{2}\underbrace{\lim_{t\to\infty}\frac{\ln t}{t^2}}_{=0} - \frac{1}{4}\underbrace{\lim_{t\to\infty}\frac{1}{t^2}}_{=0} + \frac{1}{4}$$

$$= \frac{1}{4}$$

$$: \frac{\infty}{\infty} \xrightarrow{\text{L'HÔ}} \lim_{t\to\infty} \frac{1/t}{2t} = \frac{1}{2}\lim_{t\to\infty}\frac{1}{t^2} = 0$$