5.2 EXERCISES

1–4 ▪ Use the Laws of Logarithms to expand the quantity.

1. $\ln \dfrac{x^3 y}{z^2}$

2. $\ln \sqrt{a(b^2 + c^2)}$

3. $\ln (uv)^{10}$

4. $\ln \dfrac{3x^2}{(x+1)^5}$

5–8 ▪ Express the given quantity as a single logarithm.

5. $2 \ln 4 - \ln 2$

6. $\ln 3 + \frac{1}{3} \ln 8$

7. $\frac{1}{2} \ln x - 5 \ln(x^2 + 1)$

8. $\ln x + a \ln y - b \ln z$

9–12 ▪ Make a rough sketch of the graph of each function. Do not use a calculator. Just use the graphs given in Figure 4 and, if necessary, the transformations of Section 1.2.

9. $y = -\ln x$

10. $y = \ln |x|$

11. $y = \ln(x + 3)$

12. $y = 1 + \ln(x - 2)$

13–30 ▪ Differentiate the function.

13. $f(x) = \sqrt{x} \ln x$

14. $f(x) = \ln(x^2 + 10)$

15. $f(\theta) = \ln(\cos \theta)$

16. $f(x) = \cos(\ln x)$

17. $f(x) = \sqrt[5]{\ln x}$

18. $f(x) = \ln \sqrt[5]{x}$

19. $g(x) = \ln \dfrac{a - x}{a + x}$

20. $h(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$

21. $f(u) = \dfrac{\ln u}{1 + \ln(2u)}$

22. $f(t) = \dfrac{1 + \ln t}{1 - \ln t}$

23. $F(t) = \ln \dfrac{(2t + 1)^3}{(3t - 1)^4}$

24. $y = \ln(x^4 \sin^2 x)$

25. $y = \ln |2 - x - 5x^2|$

26. $G(u) = \ln \sqrt{\dfrac{3u + 2}{3u - 2}}$

27. $y = \ln\left(\dfrac{x + 1}{x - 1}\right)^{3/5}$

28. $y = (\ln \tan x)^2$

29. $y = \tan[\ln(ax + b)]$

30. $y = \ln |\tan 2x|$

31–32 ▪ Find y' and y''.

31. $y = \ln \ln x$

32. $y = \dfrac{\ln x}{x^2}$

33–34 ▪ Differentiate f and find the domain of f.

33. $f(x) = \dfrac{x}{1 - \ln(x - 1)}$

34. $f(x) = \ln \ln \ln x$

35. If $f(x) = \dfrac{x}{\ln x}$, find $f'(e)$.

36. If $f(t) = t \ln(4 + 3t)$, find $f'(-1)$.

37–38 ▪ Find an equation of the tangent line to the curve at the given point.

37. $y = \sin(2 \ln x)$, $(1, 0)$

38. $y = \ln(x^3 - 7)$, $(2, 0)$

39. Find y' if $y = \ln(x^2 + y^2)$.

40. Find y' if $\ln xy = y \sin x$.

41. Find a formula for $f^{(n)}(x)$ if $f(x) = \ln(x - 1)$.

42. Find $\dfrac{d^9}{dx^9}(x^8 \ln x)$.

43–44 ▪ Use a graph to estimate the roots of the equation correct to one decimal place. Then use these estimates as the initial approximations in Newton's method to find the roots correct to six decimal places.

43. $(x - 4)^2 = \ln x$

44. $\ln(4 - x^2) = x$

45–48 ▪ Discuss the curve under the guidelines of Section 3.4.

45. $y = \ln(\sin x)$

46. $y = \ln(\tan^2 x)$

47. $y = \ln(1 + x^2)$

48. $y = \ln(x^2 - 3x + 2)$

CAS **49.** If $f(x) = \ln(2x + x \sin x)$, use the graphs of f, f', and f'' to estimate the intervals of increase and the inflection points of f on the interval $(0, 15]$.

50. Investigate the family of curves $f(x) = \ln(x^2 + c)$. What happens to the inflection points and asymptotes as c changes? Graph several members of the family to illustrate what you discover.

51–54 ▪ Use logarithmic differentiation to find the derivative of the function.

51. $y = (2x + 1)^5 (x^4 - 3)^6$

52. $y = \dfrac{(x^3 + 1)^4 \sin^2 x}{\sqrt[3]{x}}$

53. $y = \dfrac{\sin^2 x \tan^4 x}{(x^2 + 1)^2}$

54. $y = \sqrt[4]{\dfrac{x^2 + 1}{x^2 - 1}}$

55–62 ▪ Evaluate the integral.

55. $\displaystyle\int_1^2 \frac{dt}{8 - 3t}$

56. $\displaystyle\int_1^2 \frac{4 + u^2}{u^3}\, du$

57. $\displaystyle\int_1^e \frac{x^2 + x + 1}{x}\, dx$

58. $\displaystyle\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

59. $\displaystyle\int \frac{2 - x^2}{6x - x^3}\,dx$

60. $\displaystyle\int_e^6 \frac{dx}{x \ln x}$

61. $\displaystyle\int \frac{(\ln x)^2}{x}\,dx$

62. $\displaystyle\int \frac{\cos x}{2 + \sin x}\,dx$

63. Show that $\int \cot x\,dx = \ln|\sin x| + C$ by (a) differentiating the right side of the equation and (b) using the method of Example 12.

64. Find f if $f''(x) = x^{-2}$, $x > 0$, $f(1) = 0$, and $f(2) = 0$.

65. If g is the inverse function of $f(x) = 2x + \ln x$, find $g'(2)$.

66. (a) Find the linear approximation to $f(x) = \ln x$ near 1.
(b) Illustrate part (a) by graphing f and its linearization.
(c) For what values of x is the linear approximation accurate to within 0.1?

67. (a) By comparing areas, show that

$$\tfrac{1}{3} < \ln 1.5 < \tfrac{5}{12}$$

(b) Use the Midpoint Rule with $n = 10$ to estimate $\ln 1.5$.

68. Refer to Example 1.
(a) Find an equation of the tangent line to the curve $y = 1/t$ that is parallel to the secant line AD.
(b) Use part (a) to show that $\ln 2 > 0.66$.

69. By comparing areas, show that

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln n < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}$$

70. Prove the third law of logarithms. [*Hint:* Start by showing that both sides of the equation have the same derivative.]

71. For what values of m do the line $y = mx$ and the curve $y = x/(x^2 + 1)$ enclose a region? Find the area of the region.

72. Find $\lim_{x\to\infty} [\ln(2 + x) - \ln(1 + x)]$.

73. Use the definition of derivative to prove that

$$\lim_{x\to 0} \frac{\ln(1 + x)}{x} = 1$$

74. (a) Compare the rates of growth of $f(x) = x^{0.1}$ and $g(x) = \ln x$ by graphing both f and g in several viewing rectangles. When does the graph of f finally surpass the graph of g?
(b) Graph the function $h(x) = (\ln x)/x^{0.1}$ in a viewing rectangle that displays the behavior of the function as $x \to \infty$.
(c) Find a number N such that

$$\frac{\ln x}{x^{0.1}} < 0.1 \qquad \text{whenever} \qquad x > N$$

5.3 THE NATURAL EXPONENTIAL FUNCTION

Since ln is an increasing function, it is one-to-one and therefore has an inverse function, which we denote by exp. Thus, according to the definition of an inverse function,

$f^{-1}(x) = y \iff f(y) = x$

1

$$\exp(x) = y \iff \ln y = x$$

and the cancellation equations are

$f^{-1}(f(x)) = x$
$f(f^{-1}(x)) = x$

2

$$\exp(\ln x) = x \qquad \text{and} \qquad \ln(\exp x) = x$$

In particular, we have

$$\exp(0) = 1 \quad \text{since } \ln 1 = 0$$

$$\exp(1) = e \quad \text{since } \ln e = 1$$

We obtain the graph of $y = \exp x$ by reflecting the graph of $y = \ln x$ about the line $y = x$. (See Figure 1.) The domain of exp is the range of ln, that is, $(-\infty, \infty)$; the range of exp is the domain of ln, that is, $(0, \infty)$.

If r is any rational number, then the third law of logarithms gives

$$\ln(e^r) = r \ln e = r$$

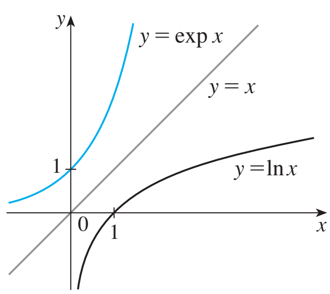

FIGURE 1