5.4 EXERCISES

1. (a) Write an equation that defines a^x when a is a positive number and x is a real number.
(b) What is the domain of the function $f(x) = a^x$?
(c) If $a \neq 1$, what is the range of this function?
(d) Sketch the general shape of the graph of the exponential function for each of the following cases.
(i) $a > 1$ (ii) $a = 1$ (iii) $0 < a < 1$

2. (a) If a is a positive number and $a \neq 1$, how is $\log_a x$ defined?
(b) What is the domain of the function $f(x) = \log_a x$?
(c) What is the range of this function?
(d) If $a > 1$, sketch the general shapes of the graphs of $y = \log_a x$ and $y = a^x$ with a common set of axes.

3–6 ▪ Write the expression as a power of e.

3. $5^{\sqrt{7}}$ **4.** 10^{x^2}

5. $(\cos x)^x$ **6.** $x^{\cos x}$

7–10 ▪ Evaluate the expression.

7. (a) $\log_{10} 1000$ (b) $\log_2 \frac{1}{16}$

8. (a) $\log_{10} 0.1$ (b) $\log_8 320 - \log_8 5$

9. (a) $\log_{12} 3 + \log_{12} 48$ (b) $\log_5 5^{\sqrt{2}}$

10. (a) $\log_a \dfrac{1}{a}$ (b) $10^{(\log_{10} 4 + \log_{10} 7)}$

11–12 ▪ Graph the given functions on a common screen. How are these graphs related?

11. $y = 2^x, \quad y = e^x, \quad y = 5^x, \quad y = 20^x$

12. $y = 3^x, \quad y = 10^x, \quad y = \left(\frac{1}{3}\right)^x, \quad y = \left(\frac{1}{10}\right)^x$

13. Use Formula 6 to evaluate each logarithm correct to six decimal places.
(a) $\log_{12} e$ (b) $\log_6 13.54$ (c) $\log_2 \pi$

14–16 ▪ Use Formula 6 to graph the given functions on a common screen. How are these graphs related?

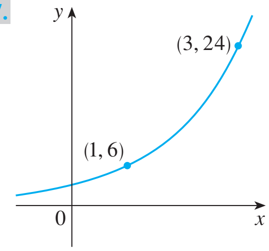
14. $y = \log_2 x, \quad y = \log_4 x, \quad y = \log_6 x, \quad y = \log_8 x$

15. $y = \log_{1.5} x, \quad y = \ln x, \quad y = \log_{10} x, \quad y = \log_{50} x$

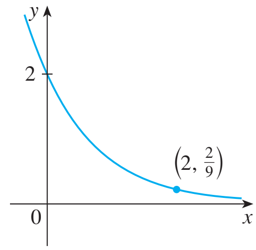
16. $y = \ln x, \quad y = \log_{10} x, \quad y = e^x, \quad y = 10^x$

17–18 ▪ Find the exponential function $f(x) = Ca^x$ whose graph is given.

17.



18.



19. (a) Show that if the graphs of $f(x) = x^2$ and $g(x) = 2^x$ are drawn on a coordinate grid where the unit of measurement is 1 inch, then at a distance 2 ft to the right of the origin the height of the graph of f is 48 ft but the height of the graph of g is about 265 mi.
(b) Suppose that the graph of $y = \log_2 x$ is drawn on a coordinate grid where the unit of measurement is an inch. How many miles to the right of the origin do we have to move before the height of the curve reaches 3 ft?

20. Compare the rates of growth of the functions $f(x) = x^5$ and $g(x) = 5^x$ by graphing both functions in several viewing rectangles. Find all points of intersection of the graphs correct to one decimal place.

21–22 ▪ Find the limit.

21. $\lim_{t \to \infty} 2^{-t^2}$ **22.** $\lim_{x \to 3^+} \log_{10}(x^2 - 5x + 6)$

23–38 ▪ Differentiate the function.

23. $h(t) = t^3 - 3^t$ **24.** $g(x) = x^4 4^x$

25. $y = 5^{-1/x}$ **26.** $y = 10^{\tan \theta}$

27. $f(u) = (2^u + 2^{-u})^{10}$ **28.** $y = 2^{3^{x^2}}$

29. $f(x) = \log_3(x^2 - 4)$ **30.** $f(x) = \log_{10}\left(\dfrac{x}{x-1}\right)$

31. $y = x^x$ **32.** $y = x^{1/x}$

33. $y = x^{\sin x}$ **34.** $y = (\sin x)^x$

35. $y = (\ln x)^x$ **36.** $y = x^{\ln x}$

37. $y = x^{e^x}$ **38.** $y = (\ln x)^{\cos x}$

39. Find an equation of the tangent line to the curve $y = 10^x$ at the point $(1, 10)$.

40. If $f(x) = x^{\cos x}$, find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

41–46 ■ Evaluate the integral.

41. $\int_1^2 10^t\, dt$

42. $\int_0^1 4^{-2u}\, du$

43. $\int \frac{\log_{10} x}{x}\, dx$

44. $\int (x^5 + 5^x)\, dx$

45. $\int 3^{\sin\theta} \cos\theta\, d\theta$

46. $\int \frac{2^x}{2^x + 1}\, dx$

47. Find the inverse function of $f(x) = 10^x/(10^x + 1)$.

48. Find y' if $x^y = y^x$.

49. Calculate $\lim_{x\to\infty} x^{-\ln x}$.

50. According to the Beer-Lambert Law, the light intensity at a depth of x meters below the surface of the ocean is $I(x) = I_0 a^x$, where I_0 is the light intensity at the surface and a is a constant such that $0 < a < 1$.
(a) Express the rate of change of $I(x)$ with respect to x in terms of $I(x)$.
(b) If $I_0 = 8$ and $a = 0.38$, find the rate of change of intensity with respect to depth at a depth of 20 m.
(c) Using the values from part (b), find the average light intensity between the surface and a depth of 20 m.

51. Prove the second law of exponents [see (3)].

52. Prove the fourth law of exponents [see (3)].

53. Deduce the following laws of logarithms from (3):
(a) $\log_a(xy) = \log_a x + \log_a y$
(b) $\log_a(x/y) = \log_a x - \log_a y$
(c) $\log_a(x^y) = y \log_a x$

54. Show that $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for any $x > 0$.

5.5 EXPONENTIAL GROWTH AND DECAY

In many natural phenomena, quantities grow or decay at a rate proportional to their size. For instance, if $y = f(t)$ is the number of individuals in a population of animals or bacteria at time t, then it seems reasonable to expect that the rate of growth $f'(t)$ is proportional to the population $f(t)$; that is, $f'(t) = kf(t)$ for some constant k. Indeed, under ideal conditions (unlimited environment, adequate nutrition, immunity to disease) the mathematical model given by the equation $f'(t) = kf(t)$ predicts what actually happens fairly accurately. Another example occurs in nuclear physics where the mass of a radioactive substance decays at a rate proportional to the mass. In chemistry, the rate of a unimolecular first-order reaction is proportional to the concentration of the substance. In finance, the value of a savings account with continuously compounded interest increases at a rate proportional to that value.

In general, if $y(t)$ is the value of a quantity y at time t and if the rate of change of y with respect to t is proportional to its size $y(t)$ at any time, then

$$\frac{dy}{dt} = ky \tag{1}$$

where k is a constant. Equation 1 is sometimes called the **law of natural growth** (if $k > 0$) or the **law of natural decay** (if $k < 0$). It is called a **differential equation** because it involves an unknown function y and its derivative dy/dt.

It's not hard to think of a solution of Equation 1. This equation asks us to find a function whose derivative is a constant multiple of itself. We have met such functions in this chapter. Any exponential function of the form $y(t) = Ce^{kt}$, where C is a constant,