The inverse hyperbolic functions are all differentiable because the hyperbolic functions are differentiable. The formulas in Table 6 can be proved either by the method for inverse functions or by differentiating Formulas 3, 4, and 5.

V EXAMPLE 4 Prove that $\dfrac{d}{dx}(\sinh^{-1}x) = \dfrac{1}{\sqrt{1 + x^2}}$.

SOLUTION Let $y = \sinh^{-1}x$. Then $\sinh y = x$. If we differentiate this equation implicitly with respect to x, we get

$$\cosh y \frac{dy}{dx} = 1$$

▪ Another method for solving Example 4 is to differentiate Formula 3.

Since $\cosh^2 y - \sinh^2 y = 1$ and $\cosh y \geqslant 0$, we have $\cosh y = \sqrt{1 + \sinh^2 y}$, so

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}$$

■

V EXAMPLE 5 Find $\dfrac{d}{dx}[\tanh^{-1}(\sin x)]$.

SOLUTION Using Table 6 and the Chain Rule, we have

$$\frac{d}{dx}[\tanh^{-1}(\sin x)] = \frac{1}{1 - (\sin x)^2}\frac{d}{dx}(\sin x)$$

$$= \frac{1}{1 - \sin^2 x}\cos x = \frac{\cos x}{\cos^2 x} = \sec x$$

■

5.7 EXERCISES

1–6 ▪ Find the numerical value of each expression.

1. (a) $\sinh 0$ (b) $\cosh 0$

2. (a) $\tanh 0$ (b) $\tanh 1$

3. (a) $\sinh(\ln 2)$ (b) $\sinh 2$

4. (a) $\cosh 3$ (b) $\cosh(\ln 3)$

5. (a) $\operatorname{sech} 0$ (b) $\cosh^{-1} 1$

6. (a) $\sinh 1$ (b) $\sinh^{-1} 1$

7–15 ▪ Prove the identity.

7. $\sinh(-x) = -\sinh x$
(This shows that sinh is an odd function.)

8. $\cosh(-x) = \cosh x$
(This shows that cosh is an even function.)

9. $\cosh x + \sinh x = e^x$

10. $\cosh x - \sinh x = e^{-x}$

11. $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

12. $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

13. $\sinh 2x = 2 \sinh x \cosh x$

14. $\dfrac{1 + \tanh x}{1 - \tanh x} = e^{2x}$

15. $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$
(n any real number)

16. If $\sinh x = \frac{3}{4}$, find the values of the other hyperbolic functions at x.

17. If $\tanh x = \frac{4}{5}$, find the values of the other hyperbolic functions at x.

18. (a) Use the graphs of sinh, cosh, and tanh in Figures 1–3 to draw the graphs of csch, sech, and coth.
(b) Check the graphs that you sketched in part (a) by using a graphing device to produce them.

19. Use the definitions of the hyperbolic functions to find each of the following limits.

(a) $\lim_{x\to\infty} \tanh x$ (b) $\lim_{x\to-\infty} \tanh x$

(c) $\lim_{x\to\infty} \sinh x$ (d) $\lim_{x\to-\infty} \sinh x$

(e) $\lim_{x\to\infty} \operatorname{sech} x$ (f) $\lim_{x\to\infty} \coth x$

(g) $\lim_{x\to0^+} \coth x$ (h) $\lim_{x\to0^-} \coth x$

(i) $\lim_{x\to-\infty} \operatorname{csch} x$

20. Prove the formulas given in Table 1 for the derivatives of the functions (a) cosh, (b) tanh, (c) csch, (d) sech, and (e) coth.

21. Give an alternative solution to Example 3 by letting $y = \sinh^{-1}x$ and then using Exercise 9 and Example 1(a) with x replaced by y.

22. Prove Equation 4.

23. Prove Formula 5 using (a) the method of Example 3 and (b) Exercise 14 with x replaced by y.

24. For each of the following functions (i) give a definition like those in (2), (ii) sketch the graph, and (iii) find a formula similar to Formula 3.

(a) csch^{-1} (b) sech^{-1} (c) $\coth^{-1}$

25. Prove the formulas given in Table 6 for the derivatives of the following functions.

(a) $\cosh^{-1}$ (b) $\tanh^{-1}$ (c) sech^{-1}

26–41 ■ Find the derivative.

26. $g(x) = \sinh^2 x$

27. $f(x) = x \cosh x$

28. $F(x) = \sinh x \tanh x$

29. $h(x) = \sinh(x^2)$

30. $f(t) = e^t \operatorname{sech} t$

31. $h(t) = \coth\sqrt{1 + t^2}$

32. $f(t) = \ln(\sinh t)$

33. $H(t) = \tanh(e^t)$

34. $y = \sinh(\cosh x)$

35. $y = e^{\cosh 3x}$

36. $y = x^2 \sinh^{-1}(2x)$

37. $y = \tanh^{-1}\sqrt{x}$

38. $y = x \tanh^{-1}x + \ln\sqrt{1 - x^2}$

39. $y = x \sinh^{-1}(x/3) - \sqrt{9 + x^2}$

40. $y = \operatorname{sech}^{-1}\sqrt{1 - x^2}, \quad x > 0$

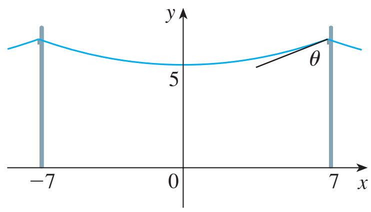
41. $y = \coth^{-1}\sqrt{x^2 + 1}$

42. A flexible cable always hangs in the shape of a catenary $y = c + a\cosh(x/a)$, where c and a are constants and $a > 0$ (see Figure 4 and Exercise 44). Graph several members of the family of functions $y = a\cosh(x/a)$. How does the graph change as a varies?

43. A telephone line hangs between two poles 14 m apart in the shape of the catenary $y = 20\cosh(x/20) - 15$, where x and y are measured in meters.

(a) Find the slope of this curve where it meets the right pole.

(b) Find the angle θ between the line and the pole.



44. Using principles from physics it can be shown that when a cable is hung between two poles, it takes the shape of a curve $y = f(x)$ that satisfies the differential equation

$$\frac{d^2y}{dx^2} = \frac{\rho g}{T}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

where ρ is the linear density of the cable, g is the acceleration due to gravity, and T is the tension in the cable at its lowest point, and the coordinate system is chosen appropriately. Verify that the function

$$y = f(x) = \frac{T}{\rho g}\cosh\left(\frac{\rho g x}{T}\right)$$

is a solution of this differential equation.

45. (a) Show that any function of the form

$$y = A \sinh mx + B \cosh mx$$

satisfies the differential equation $y'' = m^2 y$.

(b) Find $y = y(x)$ such that $y'' = 9y$, $y(0) = -4$, and $y'(0) = 6$.

46. Evaluate $\lim_{x\to\infty} \dfrac{\sinh x}{e^x}$.

47. At what point of the curve $y = \cosh x$ does the tangent have slope 1?

48. If $x = \ln(\sec\theta + \tan\theta)$, show that $\sec\theta = \cosh x$.

49. Show that if $a \neq 0$ and $b \neq 0$, then there exist numbers α and β such that $ae^x + be^{-x}$ equals either $\alpha\sinh(x + \beta)$ or $\alpha\cosh(x + \beta)$. In other words, almost every function of the form $f(x) = ae^x + be^{-x}$ is a shifted and stretched hyperbolic sine or cosine function.