In Example 6 we used l'Hospital's Rule to show that

$$\lim_{x \to 0^+} x \ln x = 0$$

Therefore

$$\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{x \ln x} = e^0 = 1$$

5.8 EXERCISES

1–36 ▪ Find the limit. Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hospital's Rule doesn't apply, explain why.

1. $\displaystyle \lim_{x \to -1} \frac{x^2 - 1}{x + 1}$

2. $\displaystyle \lim_{x \to 1} \frac{x^a - 1}{x^b - 1}$

3. $\displaystyle \lim_{x \to (\pi/2)^+} \frac{\cos x}{1 - \sin x}$

4. $\displaystyle \lim_{x \to 0} \frac{x + \tan x}{\sin x}$

5. $\displaystyle \lim_{t \to 0} \frac{e^t - 1}{t^3}$

6. $\displaystyle \lim_{t \to 0} \frac{e^{3t} - 1}{t}$

7. $\displaystyle \lim_{x \to 0} \frac{\tan px}{\tan qx}$

8. $\displaystyle \lim_{\theta \to \pi/2} \frac{1 - \sin \theta}{\csc \theta}$

9. $\displaystyle \lim_{x \to 0^+} \frac{\ln x}{x}$

10. $\displaystyle \lim_{x \to \infty} \frac{\ln \ln x}{x}$

11. $\displaystyle \lim_{t \to 0} \frac{5^t - 3^t}{t}$

12. $\displaystyle \lim_{x \to 1} \frac{\ln x}{\sin \pi x}$

13. $\displaystyle \lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$

14. $\displaystyle \lim_{x \to \infty} \frac{e^x}{x^3}$

15. $\displaystyle \lim_{x \to \infty} \frac{x}{\ln(1 + 2e^x)}$

16. $\displaystyle \lim_{x \to 0} \frac{\cos mx - \cos nx}{x^2}$

17. $\displaystyle \lim_{x \to 1} \frac{1 - x + \ln x}{1 + \cos \pi x}$

18. $\displaystyle \lim_{x \to 0} \frac{x}{\tan^{-1}(4x)}$

19. $\displaystyle \lim_{x \to 1} \frac{x^a - ax + a - 1}{(x - 1)^2}$

20. $\displaystyle \lim_{x \to 0} \frac{1 - e^{-2x}}{\sec x}$

21. $\displaystyle \lim_{x \to 0^+} \sqrt{x} \ln x$

22. $\displaystyle \lim_{x \to -\infty} x^2 e^x$

23. $\displaystyle \lim_{x \to 0} \cot 2x \sin 6x$

24. $\displaystyle \lim_{x \to 0^+} \sin x \ln x$

25. $\displaystyle \lim_{x \to \infty} x^3 e^{-x^2}$

26. $\displaystyle \lim_{x \to \infty} x \tan(1/x)$

27. $\displaystyle \lim_{x \to \infty} (xe^{1/x} - x)$

28. $\displaystyle \lim_{x \to 0} (\csc x - \cot x)$

29. $\displaystyle \lim_{x \to \infty} (x - \ln x)$

30. $\displaystyle \lim_{x \to 1} \left(\frac{1}{\ln x} - \frac{1}{x - 1} \right)$

31. $\displaystyle \lim_{x \to 0^+} x^{x^2}$

32. $\displaystyle \lim_{x \to 0^+} (\tan 2x)^x$

33. $\displaystyle \lim_{x \to 0} (1 - 2x)^{1/x}$

34. $\displaystyle \lim_{x \to \infty} \left(1 + \frac{a}{x} \right)^{bx}$

35. $\displaystyle \lim_{x \to 0^+} (\cos x)^{1/x^2}$

36. $\displaystyle \lim_{x \to \infty} x^{(\ln 2)/(1 + \ln x)}$

37–38 ▪ Use a graph to estimate the value of the limit. Then use l'Hospital's Rule to find the exact value.

37. $\displaystyle \lim_{x \to \infty} x [\ln(x + 5) - \ln x]$

38. $\displaystyle \lim_{x \to \pi/4} (\tan x)^{\tan 2x}$

39. Prove that

$$\lim_{x \to \infty} \frac{e^x}{x^n} = \infty$$

for any positive integer n. This shows that the exponential function approaches infinity faster than any power of x.

40. Prove that

$$\lim_{x \to \infty} \frac{\ln x}{x^p} = 0$$

for any number $p > 0$. This shows that the logarithmic function approaches ∞ more slowly than any power of x.

41. If an initial amount A_0 of money is invested at an interest rate r compounded n times a year, the value of the investment after t years is

$$A = A_0 \left(1 + \frac{r}{n} \right)^{nt}$$

If we let $n \to \infty$, we refer to the *continuous compounding* of interest. Use l'Hospital's Rule to show that if interest is compounded continuously, then the amount after n years is

$$A = A_0 e^{rt}$$

42. If an object with mass m is dropped from rest, one model for its speed v after t seconds, taking air resistance into account, is

$$v = \frac{mg}{c} (1 - e^{-ct/m})$$

where g is the acceleration due to gravity and c is a positive constant.

(a) Calculate $\lim_{t \to \infty} v$. What is the meaning of this limit?

(b) For fixed t, use l'Hospital's Rule to calculate $\lim_{c \to 0^+} v$. What can you conclude about the velocity of a falling object in a vacuum?

43. If an electrostatic field E acts on a liquid or a gaseous polar dielectric, the net dipole moment P per unit volume is

$$P(E) = \frac{e^E + e^{-E}}{e^E - e^{-E}} - \frac{1}{E}$$

Show that $\lim_{E \to 0^+} P(E) = 0$.

44. A metal cable has radius r and is covered by insulation, so that the distance from the center of the cable to the exterior of the insulation is R. The velocity v of an electrical impulse in the cable is

$$v = -c\left(\frac{r}{R}\right)^2 \ln\left(\frac{r}{R}\right)$$

where c is a positive constant. Find the following limits and interpret your answers.

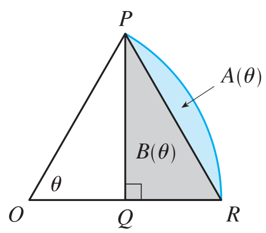
(a) $\lim_{R \to r^+} v$ (b) $\lim_{r \to 0^+} v$

45. The first appearance in print of l'Hospital's Rule was in the book *Analyse des Infiniment Petits* published by the Marquis de l'Hospital in 1696. This was the first calculus *textbook* ever published and the example that the Marquis used in that book to illustrate his rule was to find the limit of the function

$$y = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{ax^3}}$$

as x approaches a, where $a > 0$. (At that time it was common to write aa instead of a^2.) Solve this problem.

46. The figure shows a sector of a circle with central angle θ. Let $A(\theta)$ be the area of the segment between the chord PR and the arc PR. Let $B(\theta)$ be the area of the triangle PQR. Find $\lim_{\theta \to 0^+} A(\theta)/B(\theta)$.



47. If f' is continuous, $f(2) = 0$, and $f'(2) = 7$, evaluate

$$\lim_{x \to 0} \frac{f(2 + 3x) + f(2 + 5x)}{x}$$

48. For what values of a and b is the following equation true?

$$\lim_{x \to 0} \left(\frac{\sin 2x}{x^3} + a + \frac{b}{x^2}\right) = 0$$

49. If f' is continuous, use l'Hospital's Rule to show that

$$\lim_{h \to 0} \frac{f(x + h) - f(x - h)}{2h} = f'(x)$$

Explain the meaning of this equation with the aid of a diagram.

50. If f'' is continuous, show that

$$\lim_{h \to 0} \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} = f''(x)$$

51. Let

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(a) Use the definition of derivative to compute $f'(0)$.
(b) Show that f has derivatives of all orders that are defined on $\mathbb{R}$. [*Hint:* First show by induction that there is a polynomial $p_n(x)$ and a nonnegative integer k_n such that $f^{(n)}(x) = p_n(x)f(x)/x^{k_n}$ for $x \neq 0$.]

52. Let

$$f(x) = \begin{cases} |x|^x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

(a) Show that f is continuous at 0.
(b) Investigate graphically whether f is differentiable at 0 by zooming in several times toward the point (0, 1) on the graph of f.
(c) Show that f is not differentiable at 0. How can you reconcile this fact with the appearance of the graphs in part (b)?