As in Example 4, we solve this equation for the desired integral by taking the last term on the right side to the left side. Thus we have

$$n \int \sin^n x\,dx = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx$$

or
$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$ ■

The reduction formula (7) is useful because by using it repeatedly we could eventually express $\int \sin^n x\,dx$ in terms of $\int \sin x\,dx$ (if n is odd) or $\int (\sin x)^0\,dx = \int dx$ (if n is even).

6.1 EXERCISES

1–2 ▪ Evaluate the integral using integration by parts with the indicated choices of u and dv.

1. $\int x \ln x\,dx$; $\quad u = \ln x,\ dv = x\,dx$

2. $\int \theta \sec^2\theta\,d\theta$; $\quad u = \theta,\ dv = \sec^2\theta\,d\theta$

3–24 ▪ Evaluate the integral.

3. $\int x \cos 5x\,dx$

4. $\int xe^{-x}\,dx$

5. $\int re^{r/2}\,dr$

6. $\int t \sin 2t\,dt$

7. $\int x^2 \sin \pi x\,dx$

8. $\int x^2 \cos mx\,dx$

9. $\int \ln(2x+1)\,dx$

10. $\int p^5 \ln p\,dp$

11. $\int \arctan 4t\,dt$

12. $\int t^3 e^t\,dt$

13. $\int e^{2\theta} \sin 3\theta\,d\theta$

14. $\int e^{-\theta} \cos 2\theta\,d\theta$

15. $\int_0^{\pi} t \sin 3t\,dt$

16. $\int_0^1 (x^2+1)e^{-x}\,dx$

17. $\int_1^2 \frac{\ln x}{x^2}\,dx$

18. $\int_1^4 \sqrt{t}\,\ln t\,dt$

19. $\int_0^1 \frac{y}{e^{2y}}\,dy$

20. $\int_1^{\sqrt{3}} \arctan(1/x)\,dx$

21. $\int_0^{1/2} \sin^{-1}x\,dx$

22. $\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr$

23. $\int_1^2 (\ln x)^2\,dx$

24. $\int_0^t e^s \sin(t-s)\,ds$

25–28 ▪ First make a substitution and then use integration by parts to evaluate the integral.

25. $\int \sin\sqrt{x}\,dx$

26. $\int x^5 \cos(x^3)\,dx$

27. $\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2)\,d\theta$

28. $\int_1^4 e^{\sqrt{x}}\,dx$

29. (a) Use the reduction formula in Example 6 to show that

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b) Use part (a) and the reduction formula to evaluate $\int \sin^4 x\,dx$.

30. (a) Prove the reduction formula

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x \sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

(b) Use part (a) to evaluate $\int \cos^2 x\,dx$.
(c) Use parts (a) and (b) to evaluate $\int \cos^4 x\,dx$.

31. (a) Use the reduction formula in Example 6 to show that

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}x\,dx$$

where $n \geqslant 2$ is an integer.
(b) Use part (a) to evaluate $\int_0^{\pi/2} \sin^3 x\,dx$ and $\int_0^{\pi/2} \sin^5 x\,dx$.
(c) Use part (a) to show that, for odd powers of sine,

$$\int_0^{\pi/2} \sin^{2n+1}x\,dx = \frac{2\cdot 4\cdot 6\cdot \cdots \cdot 2n}{3\cdot 5\cdot 7\cdot \cdots \cdot (2n+1)}$$

32. Prove that, for even powers of sine,

$$\int_0^{\pi/2} \sin^{2n}x\,dx = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n} \frac{\pi}{2}$$

33–36 ▪ Use integration by parts to prove the reduction formula.

33. $\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$

34. $\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$

35. $\int (x^2 + a^2)^n\,dx$

$$= \frac{x(x^2+a^2)^n}{2n+1} + \frac{2na^2}{2n+1}\int (x^2+a^2)^{n-1}\,dx \quad (n \neq -\tfrac{1}{2})$$

36. $\int \sec^n x\,dx = \dfrac{\tan x \sec^{n-2}x}{n-1} + \dfrac{n-2}{n-1}\int \sec^{n-2}x\,dx$ $(n \neq 1)$

37. Use Exercise 33 to find $\int (\ln x)^3\,dx$.

38. Use Exercise 34 to find $\int x^4 e^x\,dx$.

39. Find the average value of $f(x) = x^2 \ln x$ on the interval $[1, 3]$.

40. A rocket accelerates by burning its onboard fuel, so its mass decreases with time. Suppose the initial mass of the rocket at liftoff (including its fuel) is m, the fuel is consumed at rate r, and the exhaust gases are ejected with constant velocity v_e (relative to the rocket). A model for the velocity of the rocket at time t is given by the equation

$$v(t) = -gt - v_e \ln \frac{m - rt}{m}$$

where g is the acceleration due to gravity and t is not too large. If $g = 9.8\ \mathrm{m/s^2}$, $m = 30{,}000$ kg, $r = 160$ kg/s, and $v_e = 3000$ m/s, find the height of the rocket one minute after liftoff.

41. A particle that moves along a straight line has velocity $v(t) = t^2e^{-t}$ meters per second after t seconds. How far will it travel during the first t seconds?

42. If $f(0) = g(0) = 0$ and f'' and g'' are continuous, show that

$$\int_0^a f(x)g''(x)\,dx = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x)\,dx$$

43. Suppose that $f(1) = 2$, $f(4) = 7$, $f'(1) = 5$, $f'(4) = 3$, and f'' is continuous. Find the value of $\int_1^4 xf''(x)\,dx$.

44. (a) Use integration by parts to show that

$$\int f(x)\,dx = xf(x) - \int xf'(x)\,dx$$

(b) If f and g are inverse functions and f' is continuous, prove that

$$\int_a^b f(x)\,dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)\,dy$$

[*Hint:* Use part (a) and make the substitution $y = f(x)$.]

(c) In the case where f and g are positive functions and $b > a > 0$, draw a diagram to give a geometric interpretation of part (b).

(d) Use part (b) to evaluate $\int_1^e \ln x\,dx$.

6.2 TRIGONOMETRIC INTEGRALS AND SUBSTITUTIONS

In this section we look at integrals involving trigonometric functions and integrals that can be transformed into trigonometric integrals by substitution.

TRIGONOMETRIC INTEGRALS

Here we use trigonometric identities to integrate certain combinations of trigonometric functions. We start with powers of sine and cosine.

EXAMPLE 1 Evaluate $\int \cos^3 x\,dx$.

SOLUTION Simply substituting $u = \cos x$ isn't helpful, since then $du = -\sin x\,dx$. In order to integrate powers of cosine, we would need an extra $\sin x$ factor. Similarly, a power of sine would require an extra $\cos x$ factor. Thus here we can separate one cosine factor and convert the remaining $\cos^2 x$ factor to an expression involving sine