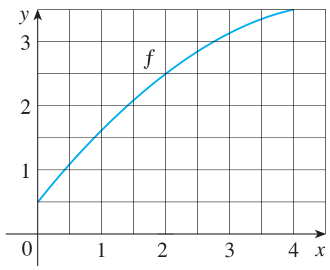
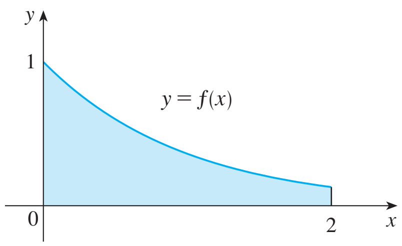
6.5 EXERCISES

1. Let $I = \int_0^4 f(x)\,dx$, where f is the function whose graph is shown.
(a) Use the graph to find L_2, R_2, and M_2.
(b) Are these underestimates or overestimates of I?
(c) Use the graph to find T_2. How does it compare with I?
(d) For any value of n, list the numbers L_n, R_n, M_n, T_n, and I in increasing order.



2. The left, right, Trapezoidal, and Midpoint Rule approximations were used to estimate $\int_0^2 f(x)\,dx$, where f is the function whose graph is shown. The estimates were 0.7811, 0.8675, 0.8632, and 0.9540, and the same number of subintervals were used in each case.
(a) Which rule produced which estimate?
(b) Between which two approximations does the true value of $\int_0^2 f(x)\,dx$ lie?



3. Estimate $\int_0^1 \cos(x^2)\,dx$ using (a) the Trapezoidal Rule and (b) the Midpoint Rule, each with $n = 4$. From a graph of the integrand, decide whether your answers are underestimates or overestimates. What can you conclude about the true value of the integral?

4. Draw the graph of $f(x) = \sin(x^2/2)$ in the viewing rectangle $[0, 1]$ by $[0, 0.5]$ and let $I = \int_0^1 f(x)\,dx$.
(a) Use the graph to decide whether L_2, R_2, M_2, and T_2 underestimate or overestimate I.
(b) For any value of n, list the numbers L_n, R_n, M_n, T_n, and I in increasing order.
(c) Compute L_5, R_5, M_5, and T_5. From the graph, which do you think gives the best estimate of I?

5–6 ▪ Use (a) the Midpoint Rule and (b) Simpson's Rule to approximate the given integral with the specified value of n. (Round your answers to six decimal places.) Compare your results to the actual value to determine the error in each approximation.

5. $\displaystyle\int_0^{\pi} x^2 \sin x\,dx, \quad n = 8$

6. $\displaystyle\int_0^1 e^{-\sqrt{x}}\,dx, \quad n = 6$

7–16 ▪ Use (a) the Trapezoidal Rule, (b) the Midpoint Rule, and (c) Simpson's Rule to approximate the given integral with the specified value of n. (Round your answers to six decimal places.)

7. $\displaystyle\int_0^2 \sqrt[4]{1 + x^2}\,dx, \quad n = 8$

8. $\displaystyle\int_0^{1/2} \sin(x^2)\,dx, \quad n = 4$

9. $\displaystyle\int_1^2 \frac{\ln x}{1 + x}\,dx, \quad n = 10$

10. $\displaystyle\int_0^3 \frac{dt}{1 + t^2 + t^4}, \quad n = 6$

11. $\displaystyle\int_0^4 e^{\sqrt{t}} \sin t\,dt, \quad n = 8$

12. $\displaystyle\int_0^4 \sqrt{1 + \sqrt{x}}\,dx, \quad n = 8$

13. $\displaystyle\int_1^5 \frac{\cos x}{x}\,dx, \quad n = 8$

14. $\displaystyle\int_4^6 \ln(x^3 + 2)\,dx, \quad n = 10$

15. $\displaystyle\int_0^3 \frac{1}{1 + y^5}\,dy, \quad n = 6$

16. $\displaystyle\int_0^1 \sqrt{z}\,e^{-z}\,dz, \quad n = 10$

17. (a) Find the approximations T_{10} and M_{10} for the integral $\int_0^2 e^{-x^2}\,dx$.
(b) Estimate the errors in the approximations of part (a).
(c) How large do we have to choose n so that the approximations T_n and M_n to the integral in part (a) are accurate to within 0.00001?

18. (a) Find the approximations T_8 and M_8 for $\int_0^1 \cos(x^2)\,dx$.
(b) Estimate the errors involved in the approximations of part (a).
(c) How large do we have to choose n so that the approximations T_n and M_n to the integral in part (a) are accurate to within 0.00001?

19. (a) Find the approximations T_{10} and S_{10} for $\int_0^1 e^x\,dx$ and the corresponding errors E_T and E_S.
(b) Compare the actual errors in part (a) with the error estimates given by (3) and (4).
(c) How large do we have to choose n so that the approximations T_n, M_n, and S_n to the integral in part (a) are accurate to within 0.00001?

20. How large should n be to guarantee that the Simpson's Rule approximation to $\int_0^1 e^{x^2}\,dx$ is accurate to within 0.00001?

CAS **21.** The trouble with the error estimates is that it is often very difficult to compute four derivatives and obtain a good upper bound K for $|f^{(4)}(x)|$ by hand. But computer algebra systems have no problem computing $f^{(4)}$ and graphing it, so we can easily find a value for K from a machine graph. This exercise deals with approximations to the integral $I = \int_0^{2\pi} f(x)\,dx$, where $f(x) = e^{\cos x}$.
(a) Use a graph to get a good upper bound for $|f''(x)|$.

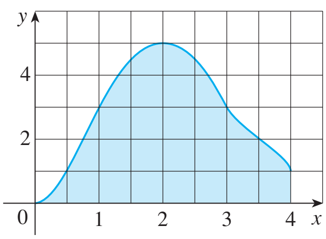
(b) Use M_{10} to approximate I.
(c) Use part (a) to estimate the error in part (b).
(d) Use the built-in numerical integration capability of your CAS to approximate I.
(e) How does the actual error compare with the error estimate in part (c)?
(f) Use a graph to get a good upper bound for $|f^{(4)}(x)|$.
(g) Use S_{10} to approximate I.
(h) Use part (f) to estimate the error in part (g).
(i) How does the actual error compare with the error estimate in part (h)?
(j) How large should n be to guarantee that the size of the error in using S_n is less than 0.0001?

CAS **22.** Repeat Exercise 21 for the integral $\int_{-1}^{1} \sqrt{4 - x^3}\, dx$.

23. Find the approximations L_n, R_n, T_n, and M_n to the integral $\int_0^1 x^3\, dx$ for $n = 4$, 8, and 16. Then compute the corresponding errors E_L, E_R, E_T, and E_M. (Round your answers to six decimal places. You may wish to use the sum command on a computer algebra system.) What observations can you make? In particular, what happens to the errors when n is doubled?

24. Find the approximations T_n, M_n, and S_n to the integral $\int_{-1}^{2} xe^x\, dx$ for $n = 6$ and 12. Then compute the corresponding errors E_T, E_M, and E_S. (Round your answers to six decimal places. You may wish to use the sum command on a computer algebra system.) What observations can you make? In particular, what happens to the errors when n is doubled?

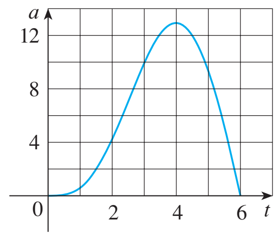
25. Estimate the area under the graph in the figure by using (a) the Trapezoidal Rule, (b) the Midpoint Rule, and (c) Simpson's Rule, each with $n = 4$.



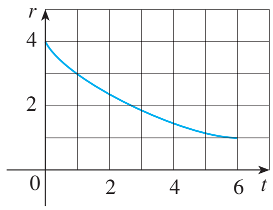
26. A radar gun was used to record the speed of a runner during the first 5 seconds of a race (see the table). Use Simpson's Rule to estimate the distance the runner covered during those 5 seconds.

t (s)	v (m/s)	t (s)	v (m/s)
0	0	3.0	10.51
0.5	4.67	3.5	10.67
1.0	7.34	4.0	10.76
1.5	8.86	4.5	10.81
2.0	9.73	5.0	10.81
2.5	10.22		

27. The graph of the acceleration $a(t)$ of a car measured in ft/s^2 is shown. Use Simpson's Rule to estimate the increase in the velocity of the car during the 6-second time interval.



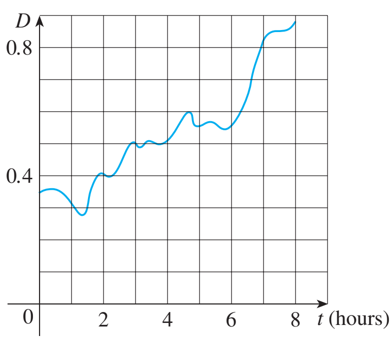
28. Water leaked from a tank at a rate of $r(t)$ liters per hour, where the graph of r is as shown. Use Simpson's Rule to estimate the total amount of water that leaked out during the first six hours.



29. The table (supplied by San Diego Gas and Electric) gives the power consumption in megawatts in San Diego County from midnight to 6:00 AM on a day in December. Use Simpson's Rule to estimate the energy used during that time period. (Use the fact that power is the derivative of energy.)

t	P	t	P
0:00	1814	3:30	1611
0:30	1735	4:00	1621
1:00	1686	4:30	1666
1:30	1646	5:00	1745
2:00	1637	5:30	1886
2:30	1609	6:00	2052
3:00	1604		

30. Shown is the graph of traffic on an Internet service provider's T1 data line from midnight to 8:00 AM. D is the data throughput, measured in megabits per second. Use Simpson's Rule to estimate the total amount of data transmitted during that time period.



31. (a) Use the Midpoint Rule and the given data to estimate the value of the integral $\int_0^{3.2} f(x)\,dx$.

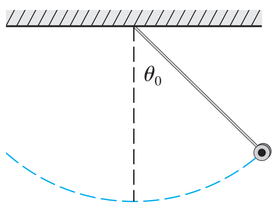
x	$f(x)$	x	$f(x)$
0.0	6.8	2.0	7.6
0.4	6.5	2.4	8.4
0.8	6.3	2.8	8.8
1.2	6.4	3.2	9.0
1.6	6.9		

(b) If it is known that $-4 \leqslant f''(x) \leqslant 1$ for all x, estimate the error involved in the approximation in part (a).

CAS **32.** The figure shows a pendulum with length L that makes a maximum angle θ_0 with the vertical. Using Newton's Second Law it can be shown that the period T (the time for one complete swing) is given by

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

where $k = \sin\left(\frac{1}{2}\theta_0\right)$ and g is the acceleration due to gravity. If $L = 1$ m and $\theta_0 = 42°$, use Simpson's Rule with $n = 10$ to find the period.



33. The intensity of light with wavelength λ traveling through a diffraction grating with N slits at an angle θ is given by $I(\theta) = N^2 \sin^2 k/k^2$, where $k = (\pi N d \sin\theta)/\lambda$ and d is the distance between adjacent slits. A helium-neon laser with wavelength $\lambda = 632.8 \times 10^{-9}$ m is emitting a narrow band of light, given by $-10^{-6} < \theta < 10^{-6}$, through a grating with 10,000 slits spaced 10^{-4} m apart. Use the Midpoint Rule with $n = 10$ to estimate the total light intensity $\int_{-10^{-6}}^{10^{-6}} I(\theta)\,d\theta$ emerging from the grating.

34. Sketch the graph of a continuous function on $[0, 2]$ for which the right endpoint approximation with $n = 2$ is more accurate than Simpson's Rule.

35. Sketch the graph of a continuous function on $[0, 2]$ for which the Trapezoidal Rule with $n = 2$ is more accurate than the Midpoint Rule.

36. Use the Trapezoidal Rule with $n = 10$ to approximate $\int_0^{20} \cos(\pi x)\,dx$. Compare your result to the actual value. Can you explain the discrepancy?

37. If f is a positive function and $f''(x) < 0$ for $a \leqslant x \leqslant b$, show that

$$T_n < \int_a^b f(x)\,dx < M_n$$

38. Show that if f is a polynomial of degree 3 or lower, then Simpson's Rule gives the exact value of $\int_a^b f(x)\,dx$.

39. Show that $\frac{1}{2}(T_n + M_n) = T_{2n}$.

40. Show that $\frac{1}{3}T_n + \frac{2}{3}M_n = S_{2n}$.

6.6 IMPROPER INTEGRALS

In defining a definite integral $\int_a^b f(x)\,dx$ we dealt with a function f defined on a finite interval $[a, b]$. In this section we extend the concept of a definite integral to the case where the interval is infinite and also to the case where f has an infinite discontinuity in $[a, b]$. In either case the integral is called an *improper* integral.

TYPE 1: INFINITE INTERVALS

Consider the infinite region S that lies under the curve $y = 1/x^2$, above the x-axis, and to the right of the line $x = 1$. You might think that, since S is infinite in extent, its area must be infinite, but let's take a closer look. The area of the part of S that lies to the left of the line $x = t$ (shaded in Figure 1) is

$$A(t) = \int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_1^t = 1 - \frac{1}{t}$$

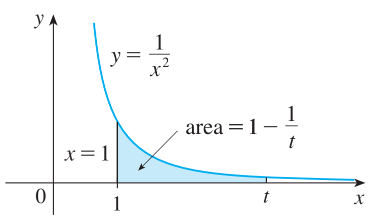


FIGURE 1

Notice that $A(t) < 1$ no matter how large t is chosen.