TABLE 2

t	$\int_1^t [(1+e^{-x})/x]\,dx$
2	0.8636306042
5	1.8276735512
10	2.5219648704
100	4.8245541204
1000	7.1271392134
10000	9.4297243064

EXAMPLE 9 The integral $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ is divergent by the Comparison Theorem because

$$\frac{1+e^{-x}}{x} > \frac{1}{x}$$

and $\int_1^\infty (1/x)\,dx$ is divergent by Example 1 [or by (2) with $p = 1$]. ■

Table 2 illustrates the divergence of the integral in Example 9. It appears that the values are not approaching any fixed number.

6.6 EXERCISES

1. Explain why each of the following integrals is improper.

(a) $\displaystyle\int_1^\infty x^4 e^{-x^4}\,dx$ (b) $\displaystyle\int_0^{\pi/2} \sec x\,dx$

(c) $\displaystyle\int_0^2 \frac{x}{x^2-5x+6}\,dx$ (d) $\displaystyle\int_{-\infty}^0 \frac{1}{x^2+5}\,dx$

2. Which of the following integrals are improper? Why?

(a) $\displaystyle\int_1^2 \frac{1}{2x-1}\,dx$ (b) $\displaystyle\int_0^1 \frac{1}{2x-1}\,dx$

(c) $\displaystyle\int_{-\infty}^\infty \frac{\sin x}{1+x^2}\,dx$ (d) $\displaystyle\int_1^2 \ln(x-1)\,dx$

3. Find the area under the curve $y = 1/x^3$ from $x = 1$ to $x = t$ and evaluate it for $t = 10$, 100, and 1000. Then find the total area under this curve for $x \geqslant 1$.

4. (a) Graph the functions $f(x) = 1/x^{1.1}$ and $g(x) = 1/x^{0.9}$ in the viewing rectangles $[0, 10]$ by $[0, 1]$ and $[0, 100]$ by $[0, 1]$.
(b) Find the areas under the graphs of f and g from $x = 1$ to $x = t$ and evaluate for $t = 10$, 100, 10^4, 10^6, 10^{10}, and 10^{20}.
(c) Find the total area under each curve for $x \geqslant 1$, if it exists.

5–32 ■ Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

5. $\displaystyle\int_1^\infty \frac{1}{(3x+1)^2}\,dx$

6. $\displaystyle\int_{-\infty}^0 \frac{1}{2x-5}\,dx$

7. $\displaystyle\int_{-\infty}^{-1} \frac{1}{\sqrt{2-w}}\,dw$

8. $\displaystyle\int_0^\infty \frac{x}{(x^2+2)^2}\,dx$

9. $\displaystyle\int_4^\infty e^{-y/2}\,dy$

10. $\displaystyle\int_{-\infty}^{-1} e^{-2t}\,dt$

11. $\displaystyle\int_{2\pi}^\infty \sin\theta\,d\theta$

12. $\displaystyle\int_{-\infty}^\infty (2-v^4)\,dv$

13. $\displaystyle\int_{-\infty}^\infty xe^{-x^2}\,dx$

14. $\displaystyle\int_{-\infty}^\infty x^2 e^{-x^3}\,dx$

15. $\displaystyle\int_0^\infty se^{-5s}\,ds$

16. $\displaystyle\int_{-\infty}^\infty \cos\pi t\,dt$

17. $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx$

18. $\displaystyle\int_{-\infty}^6 re^{r/3}\,dr$

19. $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$

20. $\displaystyle\int_1^\infty \frac{\ln x}{x^3}\,dx$

21. $\displaystyle\int_{-\infty}^\infty \frac{x^2}{9+x^6}\,dx$

22. $\displaystyle\int_0^\infty \frac{e^x}{e^{2x}+3}\,dx$

23. $\displaystyle\int_0^1 \frac{3}{x^5}\,dx$

24. $\displaystyle\int_2^3 \frac{1}{\sqrt{3-x}}\,dx$

25. $\displaystyle\int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}}$

26. $\displaystyle\int_6^8 \frac{4}{(x-6)^3}\,dx$

27. $\displaystyle\int_0^{33} (x-1)^{-1/5}\,dx$

28. $\displaystyle\int_0^1 \frac{1}{4y-1}\,dy$

29. $\displaystyle\int_{-1}^1 \frac{e^x}{e^x-1}\,dx$

30. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

31. $\displaystyle\int_0^2 z^2\ln z\,dz$

32. $\displaystyle\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$

33–38 ■ Sketch the region and find its area (if the area is finite).

33. $S = \{(x, y) \mid x \leqslant 1,\ 0 \leqslant y \leqslant e^x\}$

34. $S = \{(x, y) \mid x \geqslant -2,\ 0 \leqslant y \leqslant e^{-x/2}\}$

35. $S = \{(x, y) \mid 0 \leqslant y \leqslant 2/(x^2+9)\}$

36. $S = \{(x, y) \mid x \geqslant 0,\ 0 \leqslant y \leqslant x/(x^2+9)\}$

37. $S = \{(x, y) \mid 0 \leqslant x < \pi/2,\ 0 \leqslant y \leqslant \sec^2 x\}$

38. $S = \left\{(x, y) \mid -2 < x \leqslant 0,\ 0 \leqslant y \leqslant 1/\sqrt{x+2}\right\}$

39. (a) If $g(x) = (\sin^2 x)/x^2$, use your calculator or computer to make a table of approximate values of $\int_1^t g(x)\,dx$ for $t = 2, 5, 10, 100, 1000$, and $10{,}000$. Does it appear that $\int_1^\infty g(x)\,dx$ is convergent?
(b) Use the Comparison Theorem with $f(x) = 1/x^2$ to show that $\int_1^\infty g(x)\,dx$ is convergent.
(c) Illustrate part (b) by graphing f and g on the same screen for $1 \leqslant x \leqslant 10$. Use your graph to explain intuitively why $\int_1^\infty g(x)\,dx$ is convergent.

40. (a) If $g(x) = 1/(\sqrt{x} - 1)$, use your calculator or computer to make a table of approximate values of $\int_2^t g(x)\,dx$ for $t = 5, 10, 100, 1000$, and $10{,}000$. Does it appear that $\int_2^\infty g(x)\,dx$ is convergent or divergent?
(b) Use the Comparison Theorem with $f(x) = 1/\sqrt{x}$ to show that $\int_2^\infty g(x)\,dx$ is divergent.
(c) Illustrate part (b) by graphing f and g on the same screen for $2 \leqslant x \leqslant 20$. Use your graph to explain intuitively why $\int_2^\infty g(x)\,dx$ is divergent.

41–46 ▪ Use the Comparison Theorem to determine whether the integral is convergent or divergent.

41. $\displaystyle\int_1^\infty \frac{\cos^2 x}{1 + x^2}\,dx$

42. $\displaystyle\int_1^\infty \frac{2 + e^{-x}}{x}\,dx$

43. $\displaystyle\int_1^\infty \frac{dx}{x + e^{2x}}$

44. $\displaystyle\int_1^\infty \frac{x}{\sqrt{1 + x^6}}\,dx$

45. $\displaystyle\int_0^{\pi/2} \frac{dx}{x \sin x}$

46. $\displaystyle\int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx$

47. The integral

$$\int_0^\infty \frac{1}{\sqrt{x}\,(1 + x)}\,dx$$

is improper for two reasons: The interval $[0, \infty)$ is infinite and the integrand has an infinite discontinuity at 0. Evaluate it by expressing it as a sum of improper integrals of Type 2 and Type 1 as follows:

$$\int_0^\infty \frac{1}{\sqrt{x}\,(1 + x)}\,dx = \int_0^1 \frac{1}{\sqrt{x}\,(1 + x)}\,dx + \int_1^\infty \frac{1}{\sqrt{x}\,(1 + x)}\,dx$$

48–49 ▪ Find the values of p for which the integral converges and evaluate the integral for those values of p.

48. $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^p}\,dx$

49. $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$

50. (a) Evaluate the integral $\int_0^\infty x^n e^{-x}\,dx$ for $n = 0, 1, 2$, and 3.
(b) Guess the value of $\int_0^\infty x^n e^{-x}\,dx$ when n is an arbitrary positive integer.
(c) Prove your guess using mathematical induction.

51. (a) Show that $\int_{-\infty}^\infty x\,dx$ is divergent.
(b) Show that

$$\lim_{t\to\infty} \int_{-t}^t x\,dx = 0$$

This shows that we can't define

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{t\to\infty} \int_{-t}^t f(x)\,dx$$

52. If $\int_{-\infty}^\infty f(x)\,dx$ is convergent and a and b are real numbers, show that

$$\int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx = \int_{-\infty}^b f(x)\,dx + \int_b^\infty f(x)\,dx$$

53. A manufacturer of lightbulbs wants to produce bulbs that last about 700 hours but, of course, some bulbs burn out faster than others. Let $F(t)$ be the fraction of the company's bulbs that burn out before t hours, so $F(t)$ always lies between 0 and 1.
(a) Make a rough sketch of what you think the graph of F might look like.
(b) What is the meaning of the derivative $r(t) = F'(t)$?
(c) What is the value of $\int_0^\infty r(t)\,dt$? Why?

54. The *average speed* of molecules in an ideal gas is

$$\bar{v} = \frac{4}{\sqrt{\pi}} \left(\frac{M}{2RT}\right)^{3/2} \int_0^\infty v^3 e^{-Mv^2/(2RT)}\,dv$$

where M is the molecular weight of the gas, R is the gas constant, T is the gas temperature, and v is the molecular speed. Show that

$$\bar{v} = \sqrt{\frac{8RT}{\pi M}}$$

55. As we saw in Section 5.5, a radioactive substance decays exponentially: The mass at time t is $m(t) = m(0)e^{kt}$, where $m(0)$ is the initial mass and k is a negative constant. The *mean life* M of an atom in the substance is

$$M = -k \int_0^\infty t e^{kt}\,dt$$

For the radioactive carbon isotope, ^{14}C, used in radiocarbon dating, the value of k is -0.000121. Find the mean life of a ^{14}C atom.

56. Astronomers use a technique called *stellar stereography* to determine the density of stars in a star cluster from the observed (two-dimensional) density that can be analyzed from a photograph. Suppose that in a spherical cluster of radius R the density of stars depends only on the distance r from the center of the cluster. If the perceived star density is given by $y(s)$, where s is the observed planar distance from the center of the cluster, and $x(r)$ is the actual density, it can be shown that

$$y(s) = \int_s^R \frac{2r}{\sqrt{r^2 - s^2}}\, x(r)\, dr$$

If the actual density of stars in a cluster is $x(r) = \frac{1}{2}(R - r)^2$, find the perceived density $y(s)$.

57. Determine how large the number a has to be so that

$$\int_a^\infty \frac{1}{x^2 + 1}\, dx < 0.001$$

58. Estimate the numerical value of $\int_0^\infty e^{-x^2}\, dx$ by writing it as the sum of $\int_0^4 e^{-x^2}\, dx$ and $\int_4^\infty e^{-x^2}\, dx$. Approximate the first integral by using Simpson's Rule with $n = 8$ and show that the second integral is smaller than $\int_4^\infty e^{-4x}\, dx$, which is less than 0.0000001.

59. Show that $\int_0^\infty x^2 e^{-x^2}\, dx = \frac{1}{2}\int_0^\infty e^{-x^2}\, dx$.

60. Show that $\int_0^\infty e^{-x^2}\, dx = \int_0^1 \sqrt{-\ln y}\, dy$ by interpreting the integrals as areas.

61. Find the value of the constant C for which the integral

$$\int_0^\infty \left(\frac{1}{\sqrt{x^2 + 4}} - \frac{C}{x + 2} \right) dx$$

converges. Evaluate the integral for this value of C.

62. Find the value of the constant C for which the integral

$$\int_0^\infty \left(\frac{x}{x^2 + 1} - \frac{C}{3x + 1} \right) dx$$

converges. Evaluate the integral for this value of C.

6 REVIEW

CONCEPT CHECK

1. State the rule for integration by parts. In practice, how do you use it?

2. How do you evaluate $\int \sin^m x \cos^n x\, dx$ if m is odd? What if n is odd? What if m and n are both even?

3. If the expression $\sqrt{a^2 - x^2}$ occurs in an integral, what substitution might you try? What if $\sqrt{a^2 + x^2}$ occurs? What if $\sqrt{x^2 - a^2}$ occurs?

4. What is the form of the partial fraction expansion of a rational function $P(x)/Q(x)$ if the degree of P is less than the degree of Q and $Q(x)$ has only distinct linear factors? What if a linear factor is repeated? What if $Q(x)$ has an irreducible quadratic factor (not repeated)? What if the quadratic factor is repeated?

5. State the rules for approximating the definite integral $\int_a^b f(x)\, dx$ with the Midpoint Rule, the Trapezoidal Rule, and Simpson's Rule. Which would you expect to give the best estimate? How do you approximate the error for each rule?

6. Define the following improper integrals.

(a) $\displaystyle\int_a^\infty f(x)\, dx$ (b) $\displaystyle\int_{-\infty}^b f(x)\, dx$ (c) $\displaystyle\int_{-\infty}^\infty f(x)\, dx$

7. Define the improper integral $\int_a^b f(x)\, dx$ for each of the following cases.
(a) f has an infinite discontinuity at a.
(b) f has an infinite discontinuity at b.
(c) f has an infinite discontinuity at c, where $a < c < b$.

8. State the Comparison Theorem for improper integrals.

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. $\dfrac{x(x^2 + 4)}{x^2 - 4}$ can be put in the form $\dfrac{A}{x + 2} + \dfrac{B}{x - 2}$.

2. $\dfrac{x^2 + 4}{x(x^2 - 4)}$ can be put in the form $\dfrac{A}{x} + \dfrac{B}{x + 2} + \dfrac{C}{x - 2}$.

3. $\dfrac{x^2 + 4}{x^2(x - 4)}$ can be put in the form $\dfrac{A}{x^2} + \dfrac{B}{x - 4}$.

4. $\dfrac{x^2 - 4}{x(x^2 + 4)}$ can be put in the form $\dfrac{A}{x} + \dfrac{B}{x^2 + 4}$.

5. $\displaystyle\int_0^4 \frac{x}{x^2 - 1}\, dx = \frac{1}{2}\ln 15$

6. $\displaystyle\int_1^\infty \frac{1}{x^{\sqrt{2}}}\, dx$ is convergent.

7. If f is continuous, then $\int_{-\infty}^\infty f(x)\, dx = \lim_{t\to\infty} \int_{-t}^t f(x)\, dx$.

8. The Midpoint Rule is always more accurate than the Trapezoidal Rule.