V EXAMPLE 4 Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

SOLUTION Figure 10 shows the region and a cylindrical shell formed by rotation about the line $x = 2$. It has radius $2 - x$, circumference $2\pi(2 - x)$, and height $x - x^2$.

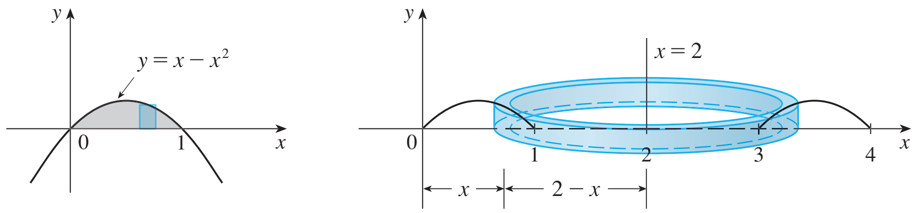


FIGURE 10

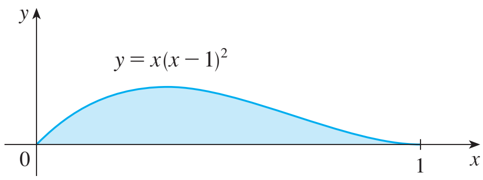
The volume of the given solid is

$$V = \int_0^1 2\pi(2 - x)(x - x^2)\, dx = 2\pi \int_0^1 (x^3 - 3x^2 + 2x)\, dx$$

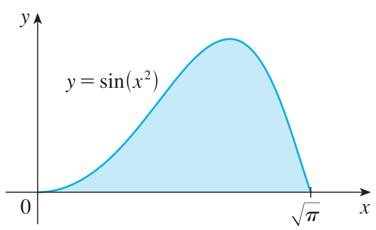
$$= 2\pi \left[\frac{x^4}{4} - x^3 + x^2 \right]_0^1 = \frac{\pi}{2}$$

7.3 EXERCISES

1. Let S be the solid obtained by rotating the region shown in the figure about the y-axis. Explain why it is awkward to use slicing to find the volume V of S. Sketch a typical approximating shell. What are its circumference and height? Use shells to find V.



2. Let S be the solid obtained by rotating the region shown in the figure about the y-axis. Sketch a typical cylindrical shell and find its circumference and height. Use shells to find the volume of S. Do you think this method is preferable to slicing? Explain.



3–7 ▪ Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the y-axis. Sketch the region and a typical shell.

3. $y = 1/x, \quad y = 0, \quad x = 1, \quad x = 2$

4. $y = x^2, \quad y = 0, \quad x = 1$

5. $y = e^{-x^2}, \quad y = 0, \quad x = 0, \quad x = 1$

6. $y = 3 + 2x - x^2, \quad x + y = 3$

7. $y = 4(x - 2)^2, \quad y = x^2 - 4x + 7$

8. Let V be the volume of the solid obtained by rotating about the y-axis the region bounded by $y = \sqrt{x}$ and $y = x^2$. Find V both by slicing and by cylindrical shells. In both cases draw a diagram to explain your method.

9–14 ▪ Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by the given curves about the x-axis. Sketch the region and a typical shell.

9. $x = 1 + y^2, \quad x = 0, \quad y = 1, \quad y = 2$

10. $x = \sqrt{y}, \quad x = 0, \quad y = 1$

11. $y = x^3, \quad y = 8, \quad x = 0$

12. $x = 4y^2 - y^3, \quad x = 0$

13. $y = 4x^2, \quad 2x + y = 6$

14. $x + y = 3, \quad x = 4 - (y - 1)^2$

15–20 ▪ Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the specified axis. Sketch the region and a typical shell.

15. $y = x^2,\ y = 0,\ x = 1,\ x = 2; \quad$ about $x = 1$

16. $y = x^2$, $y = 0$, $x = -2$, $x = -1$; about the y-axis

17. $y = x^2$, $y = 0$, $x = 1$, $x = 2$; about $x = 4$

18. $y = 4x - x^2$, $y = 8x - 2x^2$; about $x = -2$

19. $y = \sqrt{x - 1}$, $y = 0$, $x = 5$; about $y = 3$

20. $y = x^2$, $x = y^2$; about $y = -1$

21–26 ▪ Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis.

21. $y = \ln x$, $y = 0$, $x = 2$; about the y-axis

22. $y = x$, $y = 4x - x^2$; about $x = 7$

23. $y = x^4$, $y = \sin(\pi x/2)$; about $x = -1$

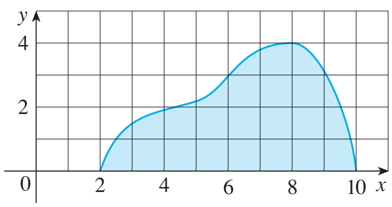
24. $y = 1/(1 + x^2)$, $y = 0$, $x = 0$, $x = 2$; about $x = 2$

25. $x = \sqrt{\sin y}$, $0 \leqslant y \leqslant \pi$, $x = 0$; about $y = 4$

26. $x^2 - y^2 = 7$, $x = 4$; about $y = 5$

27. Use the Midpoint Rule with $n = 4$ to estimate the volume obtained by rotating about the y-axis the region under the curve $y = \tan x$, $0 \leqslant x \leqslant \pi/4$.

28. (a) If the region shown in the figure is rotated about the y-axis to form a solid, use Simpson's Rule with $n = 8$ to estimate the volume of the solid.
(b) Estimate the volume if the region is rotated about the x-axis.



29–32 ▪ Each integral represents the volume of a solid. Describe the solid.

29. $\int_0^3 2\pi x^5\,dx$

30. $2\pi \int_0^2 \frac{y}{1 + y^2}\,dy$

31. $\int_0^1 2\pi(3 - y)(1 - y^2)\,dy$

32. $\int_0^{\pi/4} 2\pi(\pi - x)(\cos x - \sin x)\,dx$

33–38 ▪ The region bounded by the given curves is rotated about the specified axis. Find the volume of the resulting solid by any method.

33. $y = x^2 + x - 2$, $y = 0$; about the x-axis

34. $y = x^2 - 3x + 2$, $y = 0$; about the y-axis

35. $y = 5$, $y = x + (4/x)$; about $x = -1$

36. $x = 1 - y^4$, $x = 0$; about $x = 2$

37. $x^2 + (y - 1)^2 = 1$; about the y-axis

38. $x^2 + (y - 1)^2 = 1$; about the x-axis

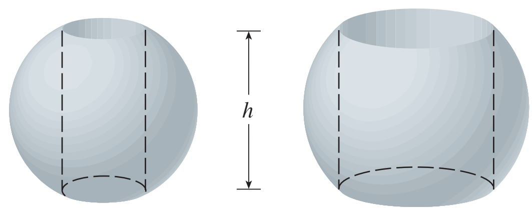
39–41 ▪ Use cylindrical shells to find the volume of the solid.

39. A sphere of radius r

40. The solid torus of Exercise 41 in Section 7.2

41. A right circular cone with height h and base radius r

42. Suppose you make napkin rings by drilling holes with different diameters through two wooden balls (which also have different diameters). You discover that both napkin rings have the same height h, as shown in the figure.
(a) Guess which ring has more wood in it.
(b) Check your guess: Use cylindrical shells to compute the volume of a napkin ring created by drilling a hole with radius r through the center of a sphere of radius R and express the answer in terms of h.



43. Use the following steps to prove Formula 2 for the case where f is one-to-one and therefore has an inverse function f^{-1}: Use the figure to show that

$$V = \pi b^2 d - \pi a^2 c - \int_c^d \pi[f^{-1}(y)]^2\,dy$$

Make the substitution $y = f(x)$ and then use integration by parts on the resulting integral to prove that

$$V = \int_a^b 2\pi x f(x)\,dx$$