Name: *ANSWER KEY* 3/10/2016

Math 202 Quiz 1

Directions Answer all questions in the space provided and box your final answers. Good luck!

1. (6 points) For each of the following three functions, state the domain and range.

$f(x) = \sin^{-1} x$

$\boxed{\text{DOM} = [-1, 1] \quad \text{RAN} = [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]}$

$g(x) = \cos^{-1} x$

$\boxed{\text{DOM} = [-1, 1] \quad \text{RAN} = [0, \pi]}$

$h(x) = \tan^{-1} x$

$\boxed{\text{DOM} = \mathbb{R} \quad \text{RAN} = (-1, 1)}$

2. (8 points) Use log rules to expand the expression $\ln\left(\frac{\sqrt[3]{xy^4}}{z^2+1}\right)$.

$$= \ln\left((xy^4)^{1/3}\right) - \ln(z^2+1)$$

$$= \frac{1}{3}\ln(xy^4) - \ln(z^2+1)$$

$$= \boxed{\frac{1}{3}\left[\ln x + 4\ln y\right] - \ln(z^2+1)}$$

3. (8 points) One cup of coffee contains 95mg of caffeine. The half-life of caffeine in the human body is 5.7 hours. How many hours after drinking one cup of coffee does the amount of caffeine in the human body fall below 10mg?

LET $A(t)$ = THE AMOUNT OF CAFFEINE IN THE HUMAN BODY t HOURS AFTER DRINKING A CUP OF COFFEE.

$$A(t) = 95\left(\tfrac{1}{2}\right)^{t/5.7}$$

SOLVE: $95\left(\tfrac{1}{2}\right)^{t/5.7} = 10$

$$\left(\tfrac{1}{2}\right)^{t/5.7} = \frac{10}{95} = \frac{2}{19}$$

$$\frac{t}{5.7}\ln\left(\tfrac{1}{2}\right) = \ln\left(\tfrac{2}{19}\right)$$

$$\boxed{t = \frac{5.7\ln\left(\frac{2}{19}\right)}{\ln\left(\frac{1}{2}\right)}} \approx 8.5 \text{ HOURS}$$

4. (8 points) Find $\dfrac{d}{dx}\cot^{-1}x$ and show your work. Answers without work will not receive full credit.

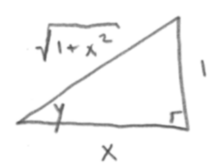
LET $y = \cot^{-1}(x)$. FIND y'.

$$\cot(y) = x \xrightarrow{\frac{d}{dx}} -\csc^2 y \cdot y' = 1$$

$$y' = \underbrace{-\frac{1}{\csc^2 y} = -\frac{1}{1+\cot^2 y}}_{1+\cot^2 y \,=\, \csc^2 y} = -\frac{1}{1+x^2}$$

OR USE THIS DIAGRAM:

$\cot(y) = x$



$\Rightarrow \csc^2 y = 1 + x^2$

5. (8 points) Differentiate the function $f(x) = x^{\ln x}$.

$$f(x) = x^{\ln x} = \left(e^{\ln x}\right)^{\ln x} = e^{(\ln x)^2}$$

$$f'(x) = e^{(\ln x)^2} \cdot 2\ln x \cdot \frac{1}{x}$$

$$= \boxed{\frac{2x^{\ln x}\ln x}{x}} \overset{\text{OR}}{=} 2x^{\ln(x)-1}\ln x$$

6. Evaluate the following integrals.

(a) (8 points) $\displaystyle\int \frac{a^x}{a^x - 1}\,dx, \quad a > 0$

Let $u = a^x - 1$

$du = a^x \ln(a)\,dx$

$\dfrac{1}{\ln(a)}\,du = a^x\,dx$

$$\rightsquigarrow \frac{1}{\ln(a)} \int \frac{1}{u}\,du = \frac{\ln(u)}{\ln(a)} + C$$

$$\rightsquigarrow \boxed{\frac{\ln(a^x - 1)}{\ln(a)} + C} = \log_a(a^x - 1) + C$$

(b) (8 points) $\displaystyle\int_{e^{\sqrt{2}/2}}^{e^{\sqrt{3}/2}} \frac{1}{x\sqrt{1 - \ln(x)^2}}\,dx$

Let $u = \ln x$

$du = \dfrac{1}{x}\,dx$

$$\rightsquigarrow \int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}(u) \Big|_{\sqrt{2}/2}^{\sqrt{3}/2}$$

$$= \frac{\pi}{3} - \frac{\pi}{4} = \boxed{\frac{\pi}{12}}$$

7. Evaluate each of the following limits.
If a limit does not exist, say so. If you apply L'ospital's Rule, say so.
(a) (8 points) $\lim_{x\to 0} e^{1/x}$

NOTICE THAT THE SIGN OF THE EXPONENT DEPENDS ON THE SIGN OF X!

$$\left.\begin{aligned} &\lim_{x\to 0^+} e^{1/x} = \infty \qquad 1/x \longrightarrow +\infty \\ &\lim_{x\to 0^-} e^{1/x} = 0 \qquad 1/x \longrightarrow -\infty \end{aligned}\right\} \therefore \boxed{\text{THE LIMIT DOES NOT EXIST}}$$

(b) (8 points) $\lim_{t\to\infty} \frac{\sinh t}{\cosh t}$

$$= \lim_{t\to\infty} \frac{\frac{e^t - e^{-t}}{2}}{\frac{e^t + e^{-t}}{2}} = \lim_{t\to\infty} \frac{e^t - e^{-t}}{e^t + e^{-t}}$$

$$= \lim_{t\to\infty} \frac{e^{-t}(e^{2t} - 1)}{e^{-t}(e^{2t} + 1)} = \lim_{t\to\infty} \frac{e^{2t} - 1}{e^{2t} + 1} \quad : \quad \frac{\infty}{\infty}$$

$$\xrightarrow{\text{L'HÔ}} \lim_{t\to\infty} \frac{2e^{2t}}{2e^{2t}} = \boxed{1}$$

8. Evaluate each of the following limits.
If a limit does not exist, say so. If you apply L'ospital's Rule, say so.

(a) (8 points) $\lim_{x\to 0^+} \sin(x)\ln(x)$ $\quad : \ 0 \cdot \infty$

$$= \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{\sin(x)}} \quad : \ \frac{\infty}{\infty}$$

$$\xrightarrow{\text{L'HÔ}} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-\cos(x)}{\sin^2(x)}} = \lim_{x\to 0^+} \frac{-\sin^2(x)}{x\cos(x)} \quad : \ \frac{0}{0}$$

$$\xrightarrow{\text{L'HÔ}} \lim_{x\to 0^+} \frac{-2\sin(x)\cos(x)}{\cos(x) - x\sin(x)} = \boxed{0}$$

(b) (8 points) $\lim_{t\to\infty} t^{(\ln 2)/(1+\ln t)}$ $\quad : \ \infty^0$

$$= \lim_{t\to\infty} \text{EXP}\left(\ln\left(t^{\ln 2/1+\ln t}\right)\right)$$

$$= \text{EXP}\left(\lim_{t\to\infty} \frac{\ln(2)\ln(t)}{1+\ln(t)}\right) \quad : \ \frac{\infty}{\infty}$$

$$\xrightarrow{\text{L'HÔ}} \text{EXP}\left(\lim_{t\to\infty} \frac{\ln(2)\cdot\frac{1}{x}}{\frac{1}{x}}\right)$$

$$= \text{EXP}(\ln(2)) = e^{\ln 2} = \boxed{2}$$