Name: *Answer Key*

Math 202 Calculus II

Due 7/23/2018

Quiz 5

Please show all work and box your final answers. If you need more room, you may use the backs of the pages. You may consult the textbook and your fellow classmates, but your work must be your own. I suggest working through the problems on scrap paper and then copying your work over to these pages. Good luck!

1. For each of the following, evaluate the integral or show that the integral is divergent.

   (a) (10 points) $\int_1^\infty \frac{\ln x}{x^2}\,dx$

   Let $u = \ln x$, $du = \frac{1}{x}\,dx$; $v = -\frac{1}{x}$, $dv = \frac{1}{x^2}\,dx$

$$= \lim_{t\to\infty} -\frac{1}{x}\ln x \Big|_1^t + \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} \frac{-\ln x}{x} - \frac{1}{x}\Big|_1^t$$

$$= \lim_{t\to\infty} \frac{-\ln t}{t} - \frac{1}{t} + \frac{\ln 1}{1} + \frac{1}{1}$$

$$= 1 - \lim_{t\to\infty} \frac{\ln t}{t} : \frac{\infty}{\infty} \overset{L'H\hat{o}}{=} 1 - \lim_{t\to\infty}\frac{1}{t} = \boxed{1}$$

   (b) (10 points) $\int_0^3 \frac{x}{x^2-1}\,dx = \underbrace{\int_0^1 \frac{x}{x^2-1}\,dx}_{1} + \underbrace{\int_1^3 \frac{x}{x^2-1}\,dx}_{2}$

1 $\lim_{t\to 1^-} \int_0^t \frac{x}{x^2-1}\,dx$ Let $u = x^2 - 1$, $du = 2x\,dx$

$$\rightsquigarrow \lim_{t\to 0^-} \frac{1}{2}\int_{-1}^{t} \frac{1}{u}\,du = \lim_{t\to 0^-} \frac{1}{2}\ln|t| - \frac{1}{2}\ln|-1| = -\infty$$

$\boxed{\text{DIVERGENT}}$

2. For each of the following, determine whether the integral converges or diverges. Justify your answer.

(a) (10 points) $\displaystyle\int_1^\infty \frac{\sqrt{x}\tan^{-1} x}{\sqrt{1+x^4}}\, dx$

$$0 \leq \frac{\sqrt{x}\ \tan^{-1} x}{\sqrt{1+x^4}} \leq \frac{\sqrt{x}\ \frac{\pi}{2}}{\sqrt{x^4}} = \frac{\pi}{2} \cdot \frac{1}{x^{3/2}} \quad \text{FOR } x \geq 1$$

SINCE $\frac{\pi}{2}\displaystyle\int_1^\infty \frac{1}{x^{3/2}}\, dx$ CONVERGES BY p-TEST (WITH $p = \frac{3}{2}$)

THE ORIGINAL INTEGRAL ALSO CONVERGES, BY COMPARISON THM.

(b) (10 points) $\displaystyle\int_1^\infty \frac{x + \sin^2 x}{x^2 - e^{-x}}\, dx$

$$\frac{x + \sin^2 x}{x^2 - e^{-x}} \geq \frac{x}{x^2} = \frac{1}{x} \geq 0 \quad \text{FOR } x \geq 1$$

SINCE $\displaystyle\int_1^\infty \frac{1}{x}\, dx$ DIVERGES BY p-TEST (WITH $p = 1$).

THE ORIGINAL INTEGRAL ALSO DIVERGES, BY COMPARISON THM.

3. Consider the region $\mathcal{R}$ bounded by the following curves.

$$x^2 = y + 1, \qquad x + y = 1$$

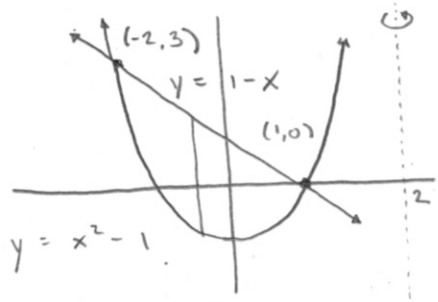
(a) (10 points) Find the volume of the solid obtained by rotating the region $\mathcal{R}$ around the line $x = 2$.

INTERSECTIONS: $x^2 - 1 = 1 - x$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$x = -2, 1$

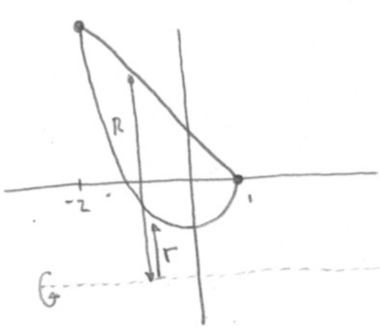


$$V = \int_{-2}^{1} 2\pi r(x) h(x)\,dx = \int_{-2}^{1} 2\pi(2-x)\underbrace{\left[(1-x)-(x^2-1)\right]}^{(2-x-x^2)} dx$$

$$= 2\pi \int_{-2}^{1} x^3 - x^2 - 4x + 4\,dx = 2\pi\left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - 2x^2 + 4x\right]_{-2}^{1}$$

$$= 2\pi\left[\frac{1}{4}(1-16) - \frac{1}{3}(1+8) - 2(1-4) + 4(1+2)\right] = \boxed{\frac{45\pi}{2}} = 22.5\pi$$

(b) (10 points) Find the volume of the solid obtained by rotating the region $\mathcal{R}$ around the line $y = -2$.



$$V = \int_{-2}^{1} \pi R^2 - \pi r^2\,dx$$

$$= \pi \int_{-2}^{1} (3-x)^2 - (x^2+1)^2\,dx$$

$R = (1-x) + 2 = 3 - x$

$r = (x^2 - 1) + 2 = x^2 + 1$

$$= \pi \int_{-2}^{1} -x^4 - x^2 - 6x + 8\,dx$$

$$= \pi\left[-\frac{1}{5}x^5 - \frac{1}{3}x^3 - 3x^2 + 8x\right]_{-2}^{1} = \pi\left[-\frac{1}{5}(1+32) - \frac{1}{3}(1+8) - 3(1-4) + 8(1+2)\right]$$

$$-\frac{33}{5} \quad - 3 \quad + 9 \quad + 24$$

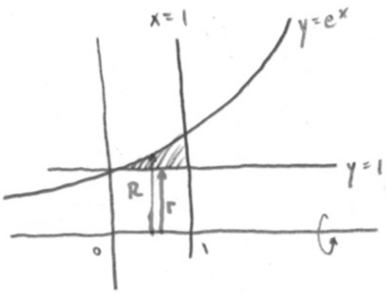
$$= \boxed{\frac{117\pi}{5}} = 23.4\pi$$

4. Consider the region $\mathcal{R}$ bounded by the following curves.

$$y = e^x, \qquad y = 1, \qquad x = 1$$

Setup two different integrals to find the volume of the solid obtained by rotating the region $\mathcal{R}$ around the x-axis in two different ways.
(Both integrals should evaluate to the same number, but you do not need to evaluate either one.)

(a) (10 points) Using the method disks/washers.



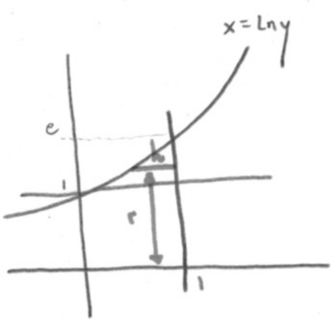
$$\boxed{V = \int_0^1 \pi (e^x)^2 - \pi (1)^2 \, dx}$$

$$= \pi \int_0^1 e^{2x} - 1 \, dx$$

$R = e^x$

$r = 1$

(b) (10 points) Using the method of cylindrical shells.



$$\boxed{V = \int_1^e 2\pi y (1 - \ln y) \, dy}$$

$$= 2\pi \int_1^e y (1 - \ln y) \, dy$$

$r = y$

$h = 1 - \ln y$

5. A curve is described parametrically by the following equations.

$$x = 1 + \frac{t^2}{2} - \ln t, \qquad y = 4t - 1$$

(a) (10 points) Find all points on the curve (if any) where the tangent line is horizontal or vertical.

$y' = 4 \neq 0 \implies$ NO HORIZONTAL TANGENTS

$x' = t - \frac{1}{t} = 0 \implies t^2 - 1 = 0 \implies t = \pm 1$, BUT -1 IS NOT IN DOMAIN.

$\therefore$ VERTICAL TANGENT AT $(x(1), y(1)) = \boxed{\left(\frac{3}{2}, 3\right)}$ $(t = 1)$

(b) (10 points) Find the exact length of the curve for $1 \le t \le e$.

$$L = \int_1^e \sqrt{\left(t - \tfrac{1}{t}\right)^2 + (4)^2}\, dt$$

THIS IS NOT SOLVEABLE BY ELEMENTARY METHODS. SORRY!

THE EQ'S SHOULD HAVE BEEN: $x = 1 + \frac{t^2}{2} - \ln t$, $y = 2t - 1$

$x' = t - \frac{1}{t}$, $y' = 2$

$$L = \int_1^e \sqrt{\left(t - \tfrac{1}{t}\right)^2 + (2)^2}\, dt = \int_1^e \sqrt{t^2 - 2 + \tfrac{1}{t^2} + 4}\, dt$$

$$= \int_1^e \sqrt{\left(t + \tfrac{1}{t}\right)^2}\, dt = \frac{1}{2}t^2 + \ln t \Big|_1^e = \boxed{\frac{1}{2}(e^2 - 1) + 1}$$