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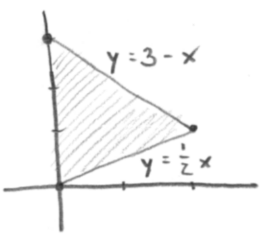
§12.4 Applications of Double Integrals

2. $$\iint_D \sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^1 r^2\,dr\,d\theta = 2\pi\cdot\frac{1}{3}r^3\Big|_0^1 = \boxed{\frac{2\pi}{3}}$$

3. $$m = \int_1^4\int_1^3 ky^2\,dx\,dy = 2K\int_1^4 y^2\,dy = \frac{2K}{3}y^3\Big|_1^4 = \frac{2K}{3}(64-1) = \boxed{42K}$$

← MASS
CENTER OF MASS
ON p. 2

5.



$$\int_0^2\int_{\frac{1}{2}x}^{3-x} x+y\,dy\,dx = \int_0^2 xy+\frac{1}{2}y^2\Big|_{\frac{1}{2}x}^{3-x}\,dx$$

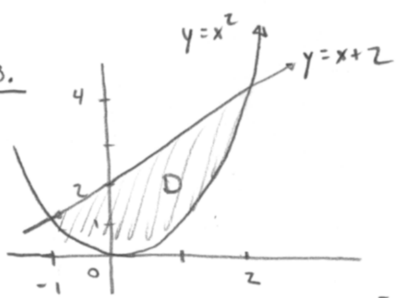
$$= \int_0^2 x(3-x)+\frac{1}{2}(3-x)^2-\frac{1}{2}x^2-\frac{1}{2}\left(\frac{1}{2}x\right)^2\,dx$$

$$\cancel{3x}-x^2+\frac{9}{2}-\cancel{3x}+\cancel{\frac{1}{2}x^2}-\cancel{\frac{1}{2}x^2}-\frac{1}{8}x^2$$

$$= \int_0^2 \frac{9}{2}-\frac{9}{8}x^2\,dx = \frac{9}{2}x-\frac{3}{8}x^3\Big|_0^2 = 9-3 = \boxed{6}$$

← MASS
CENTER OF MASS
ON p. 2

8.



$$\int_{-1}^{2}\int_{x^2}^{x+2} kx\,dy\,dx = \int_{-1}^{2}\frac{K}{2}\left[(x+2)^2-(x^2)^2\right]dx$$

$$x^2+4x+4-x^4$$

$$= \frac{K}{2}\int_{-1}^{2} x^2+4x+4-x^4\,dx = \frac{K}{2}\left[\frac{1}{3}x^3+2x^2+4x-\frac{1}{5}x^5\right]_{-1}^{2}$$

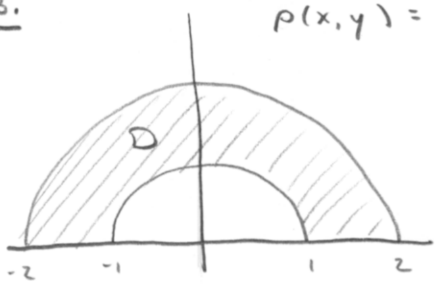
$$= \frac{K}{2}\left[\frac{1}{3}(8+1)+2(4-1)+4(2+1)-\frac{1}{5}(32+1)\right] = \frac{K}{2}\left(3+6+12-\frac{33}{5}\right)$$

$$= \boxed{\frac{36K}{5}}$$

← MASS
CENTER OF MASS
ON p. 3

13.

$$\rho(x,y) = k\sqrt{x^2+y^2} \rightsquigarrow kr$$



$$m = \int_0^{\pi}\int_1^2 kr^2\,dr\,d\theta = \frac{k\pi}{3}(2^3-1^3) = \boxed{\frac{7k\pi}{3}}$$

$$\bar{x} = \frac{3}{7k\pi}\int_0^{\pi}\int_1^2 kr^3\cos\theta\,dr\,d\theta = 0$$

$$\bar{y} = \frac{3}{7k\pi}\int_0^{\pi}\int_1^2 kr^3\sin\theta\,dr\,d\theta = \frac{45}{28k\pi}\underbrace{\int_0^{\pi}\sin\theta\,d\theta}_{2} = \boxed{\frac{45}{14k\pi}}$$

$$\frac{1}{4}r^4\Big|_1^2 = \frac{15}{4}$$

3. CONTINUED CENTER OF MASS:

$$\bar{x} = \frac{1}{42k}\int_1^4\int_1^3 kxy^2\,dx\,dy = \frac{1}{42}\cdot\frac{1}{2}x^2\Big|_1^3\int_1^4 y^2\,dy = \frac{2}{21}\cdot\frac{1}{3}y^3\Big|_1^4 = \boxed{2}$$

$$\bar{y} = \frac{1}{42k}\int_1^4\int_1^3 ky^3\,dx\,dy = \frac{1}{42}\cdot 2\cdot\frac{1}{4}y^4\Big|_1^4 = \boxed{\frac{255}{84}}$$

5. CONTINUED CENTER OF MASS:

$$\bar{x} = \frac{1}{6}\int_0^2\int_{\frac{1}{2}x}^{3-x} x^2+xy\,dy\,dx = \frac{1}{6}\int_0^2 x^2y+\frac{1}{2}xy^2\Big|_{y=\frac{1}{2}x}^{y=3-x}dx$$

$$x^2\left(3-\frac{3}{2}x\right)+\frac{1}{2}x\left(9-6x+x^2-\frac{1}{4}x^2\right)$$

$$\cancel{3x^2}-\frac{3}{2}x^3+\frac{9}{2}x-\cancel{3x^2}+\frac{1}{2}x^3-\frac{1}{8}x^3$$

$$= \frac{1}{6}\int_0^2 -\frac{9}{8}x^3+\frac{9}{2}x\,dx = \frac{1}{6}\left(-\frac{9}{32}x^4+\frac{9}{4}x^2\right)\Big|_0^2 = \frac{1}{6}\left(-\frac{9}{2}+9\right) = \boxed{\frac{3}{4}}$$

$$\bar{y} = \frac{1}{6}\int_0^2\int_{\frac{1}{2}x}^{3-x} xy+y^2\,dy\,dx = \frac{1}{6}\int_0^2 \frac{1}{2}xy^2+\frac{1}{3}y^3 \Big|_{y=\frac{1}{2}x}^{y=3-x} dx$$

$$= \frac{1}{6}\int_0^2 \frac{1}{2}x\left[(3-x)^2-\left(\tfrac{1}{2}x\right)^2\right] + \frac{1}{3}\left[(3-x)^3-\left(\tfrac{1}{2}x\right)^3\right] dx$$

$9-6x+x^2-\frac{1}{4}x^2$ $\quad$ $27-27x+9x^2-x^3-\frac{1}{8}x^3$

$\frac{9}{2}x - \cancel{3x^2} + \cancel{\frac{3}{8}x^3} + 9 - 9x + \cancel{3x^2} - \cancel{\frac{3}{8}x^3}$

$$\frac{1}{6}\int_0^2 \frac{9}{2}x + 9\,dx = \frac{1}{6}\left(\frac{9}{4}x^2+9x\right)\Big|_0^2 = \frac{1}{6}(9+18) = \boxed{\frac{9}{2}}$$

8. CONTINUED $\quad$ CENTER OF MASS

$$\bar{x} = \frac{5}{36k}\int_{-1}^2\int_{x^2}^{x+2} kx^2\,dy\,dx = \frac{5}{36}\int_{-1}^2 x^3+2x^2-x^4\,dx = \frac{5}{36}\left(\frac{1}{4}x^4+\frac{2}{3}x^3-\frac{1}{5}x^5\right)\Big|_{-1}^2$$

$x^2(x+2-x^2)$

$$= \frac{5}{36}\left(\frac{15}{4}+6-\frac{33}{5}\right) = \frac{\cancel{5}}{36}\cdot\frac{75+120-33}{\cancel{20}_4} = \frac{162}{144} = \frac{81}{72} = \frac{27}{24} = \boxed{\frac{9}{8}}$$

15.

$\rho(x,y) = x^2+y^2$

$$m = \int_0^a\int_0^{a-x} x^2+y^2\,dy\,dx = \int_0^a x^2y+\frac{1}{3}y^3\Big|_0^{a-x} dx$$

$x^2(a-x)+\frac{1}{3}(a-x)^3$

$ax^2 - x^3 + \frac{1}{3}a^3 - a^2x + ax^2 - \frac{1}{3}x^3$

$$= \int_0^a -\frac{4}{3}x^3+2ax^2-a^2x+\frac{1}{3}a^3\,dx = -\frac{1}{3}x^4+\frac{2}{3}ax^3-\frac{1}{2}a^2x^2+\frac{1}{3}a^3x\Big|_0^a$$

$$= -\cancel{\frac{1}{3}a^4}+\frac{2}{3}a^4-\frac{1}{2}a^4+\cancel{\frac{1}{3}a^4} = \boxed{\frac{a^4}{6}}$$

BY SYMMETRY, $\bar{x} = \bar{y} = \frac{6}{a^4} \int_0^a \int_0^{a-x} x^3 + xy^2 \, dy \, dx = \frac{6}{a^4} \int_0^a x^3 y + \frac{1}{3} x y^3 \Big|_{y=0}^{y=a-x} dx$

$x^3(a-x) + \frac{1}{3} x (a-x)^3$

$a x^3 - x^4 + \frac{1}{3} x (a^3 - 3a^2 x + 3 a x^2 - x^3)$

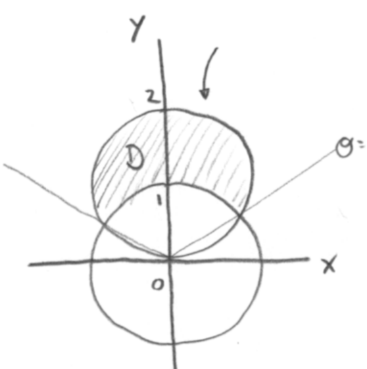
$a x^3 - x^4 + \frac{1}{3} a^3 x - a^2 x^2 + a x^3 - \frac{1}{3} x^4$

$= \frac{6}{a^4} \int_0^a -\frac{4}{3} x^4 + 2 a x^3 - a^2 x^2 + \frac{1}{3} a^3 x \, dx$

$= \frac{6}{a^4} \left(-\frac{4}{15} a^5 + \frac{1}{2} a^5 - \frac{1}{3} a^5 + \frac{1}{6} a^5 \right)$

$= 6a \cdot \frac{-8 + 15 - 10 + 5}{30} = \frac{12a}{30} = \boxed{\frac{2a}{5}}$

16. $x^2 + y^2 = 2y \rightarrow x^2 + (y-1)^2 = 1$. $\rho(x,y) = \frac{K}{\sqrt{x^2+y^2}}$



$m = \iint_D \frac{K}{\sqrt{x^2+y^2}} \, dA = \int_{\pi/6}^{5\pi/6} \int_1^2 \frac{K}{r} r \, dr \, d\theta = \boxed{\frac{2\pi}{3} K}$

$\bar{x} = \frac{3}{2K\pi} \int_{\pi/6}^{5\pi/6} \int_1^2 K r \cos\theta \, dr \, d\theta = \frac{3}{2K\pi} \cdot \frac{3}{2} \int_{\pi/6}^{5\pi/6} \cos\theta \, d\theta = \boxed{0}$

$\bar{y} = \frac{3}{2K\pi} \int_{\pi/6}^{5\pi/6} \int_1^2 K r \sin\theta \, dr \, d\theta = \frac{9}{4\pi} \int_{\pi/6}^{5\pi/6} \sin\theta \, d\theta = -\frac{9}{4\pi} \cos\theta \Big|_{\pi/6}^{5\pi/6}$

$= \frac{9}{2\pi} \cdot \frac{\sqrt{3}}{2} = \boxed{\frac{9\sqrt{3}}{4\pi}}$