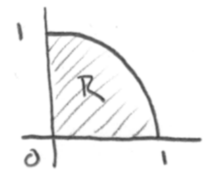
Name: *ANSWER KEY* 12/22/2014

Math 203 Calculus III **Quiz 4**

Please show all work and **box your final answers**. If you need more room, you may use the backs of the pages. Calculators are not allowed. Good luck!

1. (8 points) Find the center of mass of the lamina which occupies the portion of the unit disk $x^2 + y^2 = 1$ which lies in the first quadrant and has density $\rho(x, y) = xy$.

$$m = \iint_R \rho(x,y)\, dA = \int_0^{\pi/2}\int_0^1 \rho(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$



$$= \int_0^{\pi/2}\int_0^1 r^3 \cos\theta \sin\theta\, dr\, d\theta = \frac{1}{4}\int_0^{\pi/2} \cos\theta \sin\theta\, d\theta$$

LET $u = \sin\theta$

$du = \cos\theta\, d\theta$ $\rightsquigarrow \frac{1}{4}\int_0^1 u\, du = \frac{1}{8}$

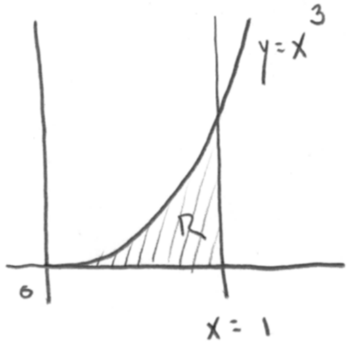
$$\bar{x} = \frac{1}{m}\iint_R x\rho(x,y)\, dA = 8\int_0^{\pi/2}\int_0^1 r^4 \cos^2\theta \sin\theta\, d\theta = \frac{8}{5}\int_0^{\pi/2} \cos^2\theta \sin\theta\, d\theta$$

LET $u = \cos\theta\, d\theta$

$-du = \sin\theta\, d\theta$ $\rightsquigarrow \frac{8}{5}\int_0^1 u^2\, du = \frac{8}{15}$

$\bar{y} = \bar{x} = \frac{8}{15}$ BY SYMMETRY. $(\bar{x}, \bar{y}) = \boxed{\left(\frac{8}{15}, \frac{8}{15}\right)}$

2. (8 points) Find the area of the portion of the surface $z = x^3 + y$ that lies above the region in the xy-plane bounded by $y = x^3$, $x = 1$, and the x-axis.



$$SA = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA$$

$$= \int_0^1 \int_0^{x^3} \sqrt{1 + (3x^2)^2 + (1)^2}\, dy\, dx$$

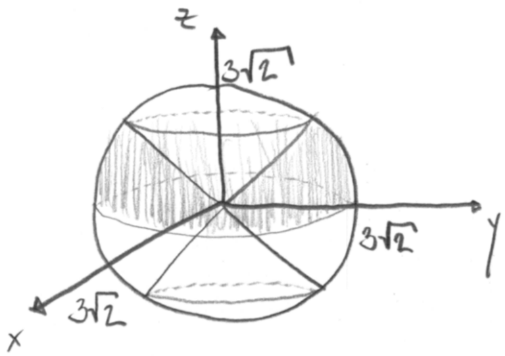
$$= \int_0^1 x^3 \sqrt{9x^4 + 2}\, dx$$

Let $u = 9x^4 + 2$

$du = 36x^3\, dx$

$$\rightsquigarrow \frac{1}{36} \int_2^{11} \sqrt{u}\, du = \frac{1}{54} \left[u^{3/2} \right]_2^{11} = \boxed{\frac{11^{3/2} - 2^{3/2}}{54}}$$

3. (8 points) Find the volume of the solid that lies inside the sphere $x^2 + y^2 + z^2 = 18$, outside the cone $z^2 = x^2 + y^2$, and above the xy-plane.



$$V = \iiint_S 1 \, dV = \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{3\sqrt{2}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 2\pi \frac{(3\sqrt{2})^3}{3} \int_{\pi/4}^{\pi/2} \sin\phi \, d\phi$$

$$= 36\sqrt{2}\,\pi \cos\phi \Big|_{\pi/2}^{\pi/4} = \boxed{36\pi}$$

4. (4 points) Is the series $\sum_{n=1}^{\infty} 2^{2n}5^{1-n}$ convergent or divergent? If it is convergent, find its sum.

$$= \sum_{n=1}^{\infty} \frac{4^n}{5^{n-1}} = 4 \sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^{n-1} = \frac{4}{1-\frac{4}{5}}$$

$$= 4 \cdot 5 = \boxed{20} \quad \text{(CONVERGES)}$$

5. (4 points) Does the series $\sum_{n=1}^{\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}}$ converge or diverge? Show work to support your answer.

SINCE $\dfrac{n+5}{\sqrt[3]{n^7+n^2}} \leq \dfrac{n+5}{\sqrt[3]{n^7}} = \dfrac{1}{n^{4/3}} + \dfrac{5}{n^{7/3}}$

WE HAVE $\displaystyle\sum_{n=1}^{\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{4/3}} + 5 \sum_{n=1}^{\infty} \frac{1}{n^{7/3}}$

CONVERGES BY p - TEST

$\therefore$ THE SERIES $\boxed{\text{CONVERGES BY COMPARISON THEOREM.}}$

6. Determine whether each of the following series is absolutely convergent, conditionally convergent, of divergent.

(a) (4 points) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\sqrt{\ln n}}$

$$\lim_{n\to\infty} \left| \frac{(-1)^n}{n\sqrt{\ln(n)}} \right| = 0 \Rightarrow \text{CONVERGES BY ALT. SERIES TEST}$$

HOWEVER $\int_2^{\infty} \frac{1}{n\sqrt{\ln(n)}}\, dn$ LET $u = \ln(n)$, $du = \frac{1}{n}\, dn$

$$\sim \int_{\ln 2}^{\infty} u^{-1/2}\, du = 2u^{1/2} \Big|_{\ln 2}^{\infty} \longrightarrow \infty$$

(DOES NOT CONVERGE ABSOLUTELY)

$\therefore$ $\boxed{\text{CONVERGES CONDITIONALLY}}$

(b) (4 points) $\sum_{n=1}^{\infty} \frac{5(-3)^n}{2^{2n}}$

$$= 5 \underbrace{\sum_{n=1}^{\infty} \left(\frac{-3}{4}\right)^n}_{\text{GEOMETRIC SERIES}} = \frac{5}{1 + \frac{3}{4}} = \boxed{\frac{20}{7}}$$