Name: *Answer Key* 11/29/2018

Math 20300-ST Calculus III Quiz 4

Please show all work and **box your final answers**. If you need more room, you may use the backs of the pages. Calculators are not allowed. Good luck!

1. (10 points) Find all critical points of the following function and classify each as either a local maximum, local minimum, or saddle point.

$$f(x,y) = x^2y + 2x^2 - y^2 - 6y$$

$f_x = 2xy + 4x = 0$

$2x(y+2) = 0$

$x = 0$ or $y = -2$

$f_y = x^2 - 2y - 6 = 0$

IF $x = 0$: $-2y - 6 = 0$

$y = -3$

IF $y = -2$: $x^2 - 2(-2) - 6 = 0$

$x^2 - 2 = 0$

$x = \pm\sqrt{2}$

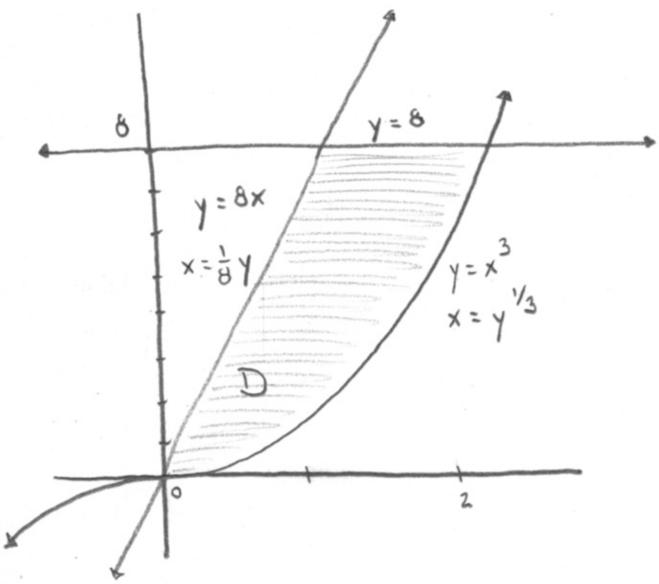
CRITICAL POINTS: $(0,-3)$, $(\sqrt{2},-2)$, $(-\sqrt{2},-2)$

$f_{xx} = 2y+4$ $f_{yy} = -2$ $f_{xy} = f_{yx} = 2x$

$D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - f_{xy}(x,y)^2$ (2^{nd} DERIV. TEST)

- $(0,-3)$: $D = (-2)(-2) - 0 > 0$, $f_{xx} = -2 < 0 \Rightarrow \boxed{(0,-3) \text{ LOCAL MAX}}$
- $(\sqrt{2},-2)$: $D = (0)(-2) - (2\sqrt{2})^2 < 0 \Rightarrow \boxed{(\sqrt{2},-2) \text{ SADDLE POINT}}$
- $(-\sqrt{2},-2)$: $D = (0)(-2) - (-2\sqrt{2})^2 < 0 \Rightarrow \boxed{(-\sqrt{2},-2) \text{ SADDLE POINT}}$

2. (10 points) Evaluate $\iint_D x^2y\,dA$, where D is the first-quadrant region bounded by $y = x^3$, $y = 8x$, and $y = 8$.



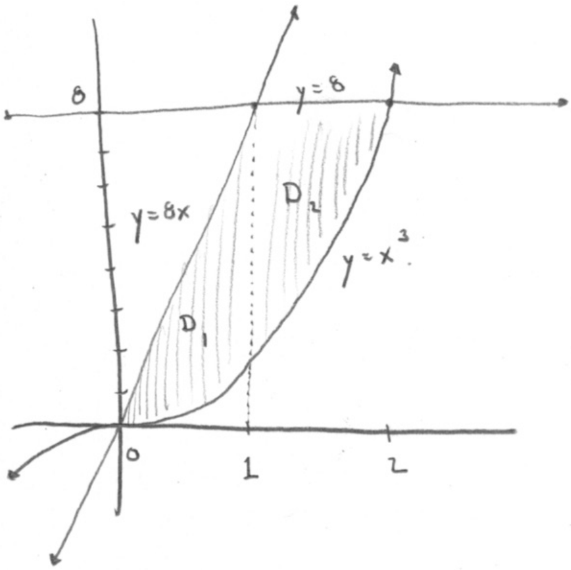
$$\text{TYPE II}: \quad \int_0^8 \int_{\frac{1}{8}y}^{y^{1/3}} x^2 y \, dx\, dy = \frac{1}{3}\int_0^8 \left(\left(y^{1/3}\right)^3 - \left(\tfrac{1}{8}y\right)^3 \right) y\, dy$$

$$= \frac{1}{3}\int_0^8 y^2 - \frac{1}{8^3}y^4 \, dy = \frac{1}{3}\left[\frac{1}{3}y^3 - \frac{1}{8^3\cdot 5}y^5 \right]_0^8$$

$$= \frac{1}{3}\left(\frac{8^3}{3} - \frac{8^5}{8^3\cdot 5} \right) = \frac{8^2}{3}\left(\frac{8}{3} - \frac{1}{5} \right) = \boxed{\frac{64}{3}\left(\frac{37}{15}\right)}$$

$$= \frac{2368}{45} = 52.6\overline{2}$$

2. (10 points) Evaluate $\iint_D x^2y\,dA$, where D is the first-quadrant region bounded by $y = x^3$, $y = 8x$, and $y = 8$.



TYPE I :

$$\iint_D x^2y\,dA = \iint_{D_1} x^2y\,dA + \iint_{D_2} x^2y\,dA$$

$$= \int_0^1\int_{x^3}^{8x} x^2y\,dy\,dx + \int_1^2\int_{x^3}^{8} x^2y\,dy\,dx$$

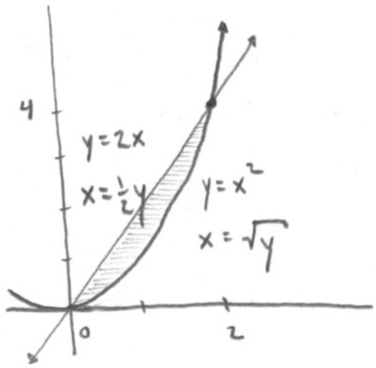
$$= \frac{1}{2}\int_0^1 64x^4 - x^8\,dx + \frac{1}{2}\int_1^2 64x^2 - x^8\,dx$$

$$= \frac{1}{2}\left[\frac{64}{5}x^5 - \frac{1}{9}x^9\right]_0^1 + \frac{1}{2}\left[\frac{64}{3}x^3 - \frac{1}{9}x^9\right]_1^2$$

$$= \frac{32}{5} - \cancel{\frac{1}{18}} + \frac{256}{3} - \frac{256}{9} - \left(\frac{32}{3} - \cancel{\frac{1}{18}}\right) = \frac{2368}{45} = 52.6\overline{2}$$

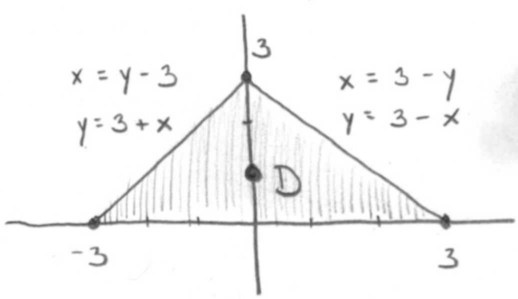
3. (10 points) Using Fubini's theorem, reverse the order of integration in the following integral, rewriting the bounds of integration as necessary. Note that you cannot evaluate the integral since the function f is not specified.

$$\int_0^2 \int_{x^2}^{2x} f(x, y)\, dy\, dx$$



$$\boxed{\int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} f(x,y)\, dx\, dy}$$

4. (10 points) A triangular lamina with uniform density $\rho = 1$ has vertices at the points $(-3, 0)$, $(0, 3)$, and $(3, 0)$. Compute the y-coordinate of the center of gravity, $\bar{y}$. You may be able to compute the mass without integrating, but you may use integration if you wish.



AREA $= \frac{1}{2} bh = \frac{1}{2}(6)(3) = 9$

MASS $M =$ DENSITY × AREA $= 1 \cdot 9 = 9$

$\bar{y} = \dfrac{M_x}{M}$, $M_x = \iint_D y \rho \, dA$

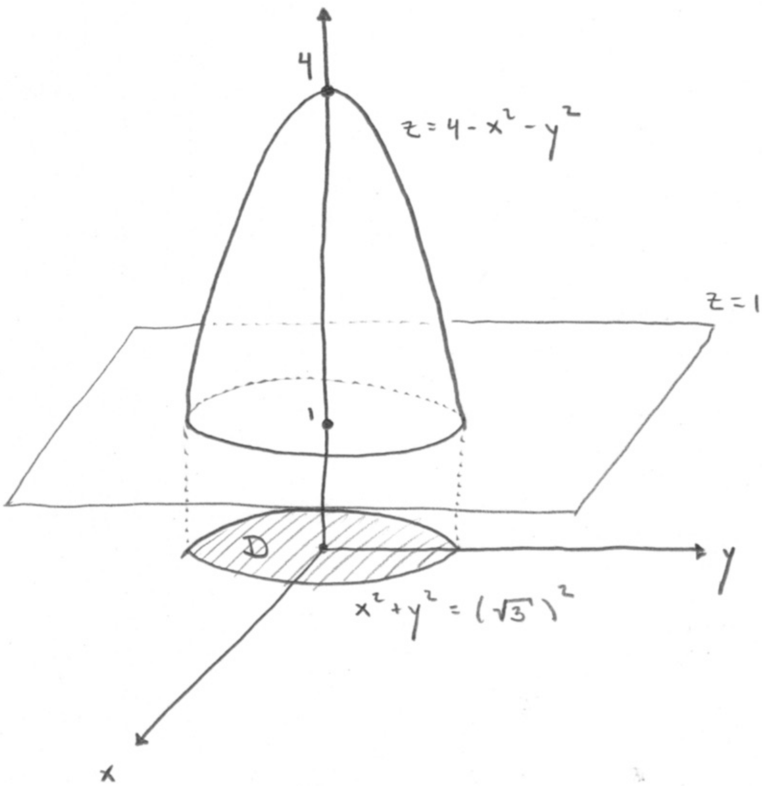
$$M_x = \int_0^3 \int_{y-3}^{3-y} y \, dx\, dy = \int_0^3 y[(3-y) - (y-3)]\, dy$$

$$= \int_0^3 6y - 2y^2 \, dy = 3y^2 - \frac{2}{3}y^3 \Big|_0^3 = 27 - 18 = 9$$

$\therefore \bar{y} = \dfrac{9}{9} = \boxed{1}$

5. (10 points) Let S be the part of the surface $z = 4 - x^2 - y^2$ that lies above the plane $z = 1$. Find the surface area of S.

INTERSECTION: $4 - x^2 - y^2 = 1 \longrightarrow x^2 + y^2 = 3, \quad z = 1.$



$$S.A. = \iint_D \sqrt{\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + 1}\, dA = \iint_D \sqrt{4x^2 + 4y^2 + 1}\, dA$$

$$(\text{POLAR COORD}) = \int_0^{2\pi}\int_0^{\sqrt{3}} \sqrt{4r^2+1}\; r\,dr\,d\theta \qquad \begin{array}{l} u = 4r^2 + 1 \\ du = 8r\,dr \longrightarrow \frac{1}{8}du = r\,dr \end{array}$$

$$= \frac{2\pi}{8}\int_1^{13} u^{1/2}\,du = \frac{\pi}{6}\, u^{3/2}\Big|_1^{13} = \boxed{\frac{\pi}{6}\left(13^{3/2} - 1\right)}$$