MATHEMATICS 205 **FINAL EXAM** **Fall, 2017**

NAME * Answer Key *

SECTION_____________

INSTRUCTOR__________________________

(1) Turn off all cell phones and put them and all notes away and out of sight.

(2) NO CALCULATORS. NO scrap paper. Use the paper provided.

(3) Leave all numerical answers in exact form. Simplify when reasonable but leave the answers in terms of π, $\sqrt{}$, e, ln and fractions.

(4) SHOW YOUR WORK. Points will be deducted for solutions which are not supported by your written work.

Part I Answer all questions in this part (60 POINTS).

(1) Find $\frac{dy}{dx}$ and simplify when reasonable. (5 points each):

(a) $y = (\ln(x))^2 + \ln(x^3)$

$$\frac{dy}{dx} = 2\ln(x) \cdot \frac{1}{x} + \frac{1}{x^3} \cdot 3x^2$$

$$\boxed{\frac{dy}{dx} = \frac{2\ln x}{x} + \frac{3}{x} = \frac{2\ln x + 3}{x}}$$

(b) $y = \frac{\sqrt{x}}{x+1}$

$$\frac{dy}{dx} = \frac{(x+1)\cdot\frac{1}{2}x^{-1/2} - x^{1/2}}{(x+1)^2} = \frac{\frac{1}{2}x^{-1/2}\left((x+1) - 2x\right)}{(x+1)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{1-x}{2\sqrt{x}\,(x+1)^2}}$$

(c) $y = xe^{\sqrt{x}}$

$$\frac{dy}{dx} = e^{\sqrt{x}} + x\,e^{\sqrt{x}}\cdot\frac{1}{2}x^{-1/2}$$

$$\boxed{\frac{dy}{dx} = e^{\sqrt{x}}\left(1 + \frac{1}{2}\sqrt{x}\right)}$$

(2) Compute each of the following integrals. (6 points each):

(a) $\int xe^{x^2} + 1\, dx$

LET $u = x^2$

$\frac{1}{2} du = x\, dx$

$= \frac{1}{2}\int e^u\, du + \int 1\, dx$

$= \frac{1}{2} e^u + x + C \quad \rightsquigarrow \quad \boxed{\frac{1}{2} e^{x^2} + x + C}$

(b) $\int \frac{x^2-2x+5}{x}\, dx$

$= \int x - 2 + \frac{5}{x} \quad dx$

$= \boxed{\frac{1}{2} x^2 - 2x + 5 \ln|x| + C}$

(c) $\int_1^e \frac{x}{x+1}\,dx$ LET $u = x + 1$ AND $x = u - 1$

$du = dx$

$$\rightsquigarrow \int_2^{e+1} \frac{u-1}{u}\,du = \int_2^{e+1} 1 - \frac{1}{u}\,du$$

$$= u - \ln|u| \Big|_2^{e+1} = \left[e+1 - \ln(e+1)\right] - \left[2 - \ln 2\right]$$

$$= \boxed{e - 1 + \ln 2 - \ln(e+1)} = e - 1 + \ln\left(\frac{2}{e+1}\right)$$

(d) $\int_1^e \frac{\ln(x)+1}{x}\,dx$ LET $u = \ln(x) + 1$

$du = \frac{1}{x}\,dx$

$$\rightsquigarrow \int_1^2 u\,du = \frac{1}{2}u^2 \Big|_1^2$$

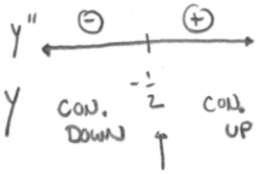
$$= \frac{1}{2}\left(2^2 - 1^2\right) = \boxed{\frac{3}{2}}$$

(3) (a)(8 points) For the curve given by the equation $y = 2x^3 + 3x^2 - 12x$, sketch the graph, labeling the relative maxima (MAX), relative minima (MIN) and points of inflection (POI). Describe where the curve is increasing and where decreasing. Describe where the curve is concave up and where concave down.

$y' = 6x^2 + 6x - 12$

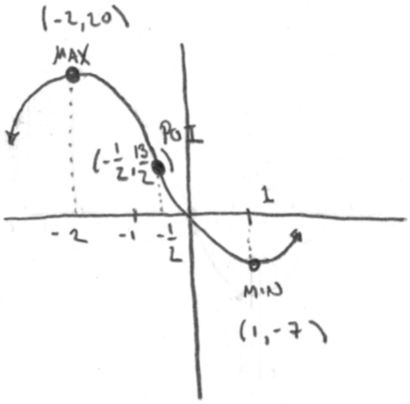
$= 6(x+2)(x-1)$

$y'' = 12x + 6$



P.O.I. $(-\frac{1}{2}, y(-\frac{1}{2}))$

$= (-\frac{1}{2}, \frac{13}{2})$



(b) (8 points) Determine that (absolute) maximum and minimum values for $f(x) = 2x^3 + 3x^2 - 12x$ on the closed interval $[-1, 2]$.

CRITICAL POINT $x = 1$ IS ONLY CRITICAL POINT IN $[-1, 2]$

x	f(x)
-1	$2(-1)^3 + 3(-1)^2 - 12(-1) = -2 + 3 + 12 = 13$ ABS. MAX. VALUE
1	$2(1)^3 + 3(1)^2 - 12(1) = 2 + 3 - 12 = -7$ ABS. MIN. VALUE.
2	$2(2)^3 + 3(2)^2 - 12(2) = 16 + 12 - 24 = 4$

$$R(t) = \overset{\substack{\text{INITIAL}\\\text{MASS}}}{\searrow} M_0\, e^{\overset{\substack{\text{REL. DECAY RATE}\\\downarrow}}{kt}}$$

$\downarrow$

(4) (5 points) Let $R(t)$ be a sample of a radioactive element with a half-life of 20 years, where the time t is measured in years. The initial sample size was 50 grams. Assume exponential decay. What is the size of the sample after 50 years? You may leave your answer in exponential form.

$$R(20) = 50\, e^{k(20)} = 25$$

$$e^{20k} = \frac{1}{2}$$

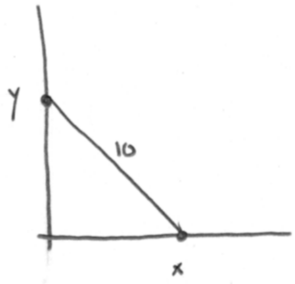
$$k = \frac{\ln(\frac{1}{2})}{20} \Rightarrow R(t) = 50\, e^{\ln(\frac{1}{2}) \cdot \frac{t}{20}}$$

$$R(t) = 50\left(\frac{1}{2}\right)^{t/20}$$

$$\boxed{R(50) = 50\left(\frac{1}{2}\right)^{5/2}}$$

PartII Answer all sections of four (4) questions out of the six (6) questions in this part (10 points each).

(5) A ladder 10 feet long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 2 feet per second, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 feet from the wall?



$$x^2 + y^2 = 10^2$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{-x \frac{dx}{dt}}{y}$$

WHEN $x = 6$, $y = 8$

$$\frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = \frac{-(6)(2)}{8} = \frac{-12}{8}$$

$$\boxed{\frac{dy}{dt} = -\frac{3}{2} \text{ ft/s}}$$

(6) (a) Use the definition of the derivative to compute the derivative of the function given by $f(x) = \sqrt{6-2x}$ and show that you answer agrees with the one given by the differentiation rules.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{6-2(x+h)} - \sqrt{6-2x}}{h}$$

$$* \quad = \lim_{h\to 0} \frac{\sqrt{6-2(x+h)} - \sqrt{6-2x}}{h} \cdot \frac{\sqrt{6-2(x+h)} + \sqrt{6-2x}}{\sqrt{6-2(x+h)} + \sqrt{6-2x}} \quad *$$

$$= \lim_{h\to 0} \frac{6-2(x+h)-(6-2x)}{h\left(\sqrt{6-2(x+h)} + \sqrt{6-2x}\right.} = \lim_{h\to 0} \frac{-2\cancel{h}}{\cancel{h}\left(\sqrt{6-2(x+h)} + \sqrt{6-2x}\right)}$$

$$\left(\text{PLUG IN } h=0\right) = \frac{-2}{2\sqrt{6-2x}} = \boxed{\frac{-1}{\sqrt{6-2x}}}$$

(b) For a curve given by $y = 2x^3 - 5x$, compute the equation of the tangent line at the point $(1, 2)$.

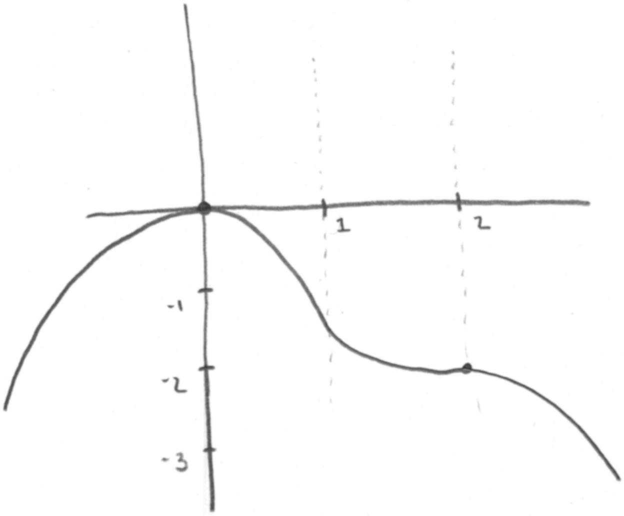
$$f(x) = 2x^3 - 5x$$

$$f'(x) = 6x^2 - 5 , \qquad f'(1) = 6-5 = 1$$

$$y - 2 = f'(1)(x-1)$$

$$\boxed{\begin{gathered} y-2 = x-1 \\ \text{or} \\ y = x+1 \end{gathered}}$$

7. (7) A function f satisfies the conditions: $f(0) = 0, f(2) = -2$, $f'(0) = f'(2) = 0$, and $f''(x) < 0$ when $x < 1$ and when $x > 2$, while $f''(x) > 0$ when $1 < x < 2$. (a) Sketch a graph for $y = f(x)$ which is consistent with the given information.



3 (b) Where is the derivative $f'(x)$ negative?

$\boxed{(0,2) \cup (2,\infty) \text{ or } (0,\infty)}$

(8) (a) Compute the limits:

3 (i) $Lim_{x\to 1}\ \frac{e^{x-1}}{2+\ln(x)}$ PLUGIN

$$= \frac{e^0}{2+\ln 1} = \boxed{\frac{1}{2}}$$

3 (ii) $Lim_{x\to 2}\frac{x^2\ -\ 4x\ +\ 4}{(x^2\ -\ 4)^2}$.

$$= \lim_{x\to 2} \frac{\cancel{(x-2)^2}}{(x+2)^2\cancel{(x-2)^2}}$$

$$= \lim_{x\to 2} \frac{1}{(x+2)^2} = \boxed{\frac{1}{16}}$$

4 (b) Use linear approximation to estimate $\sqrt{80}$.

$f(x) = \sqrt{x}$ USE TANGENT LINE AT $(81, f(81))$

$L(x) = f(81) + f'(81)(x-81)$

$f'(x) = \frac{1}{2\sqrt{x}}$, $f'(81) = \frac{1}{18}$, $f(81) = 9$

$L(x) = 9 + \frac{1}{18}(x-81)$

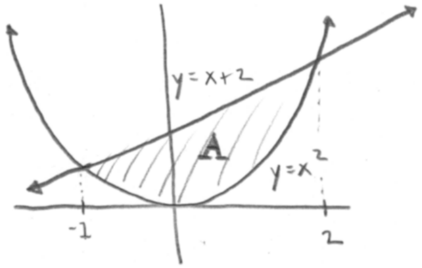
$$L(80) = 9 + \frac{1}{18}(80-81) = 9 - \frac{1}{18} = \boxed{8\tfrac{17}{18} = \frac{161}{18}}$$

7 (9) (a) Compute the area of the bounded region between the curves $y = x^2$ and $y = x + 2$. Include a sketch of the curves.

INTERSECTION(S) : $x^2 = x + 2 \rightarrow x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = -1, 2$



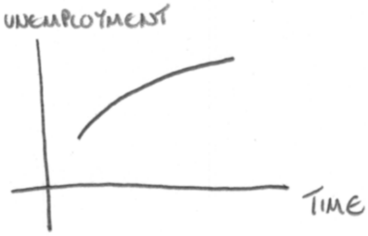
$$A = \int_{-1}^{2} (x+2) - x^2 \, dx$$

$$= \frac{1}{2}x^2 + 2x - \frac{1}{3}x^3 \Big|_{-1}^{2}$$

$$= \left[\frac{1}{2}(2)^2 + 2(2) - \frac{1}{3}(2)^3\right] - \left[\frac{1}{2}(-1)^2 + 2(-1) - \frac{1}{3}(-1)^3\right] = 2 + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}$$

$$\left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = 8 - \frac{9}{3} - \frac{1}{2} = \frac{9}{2}$$

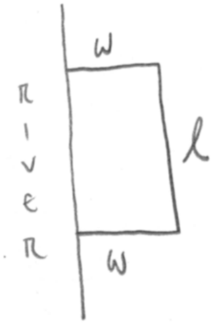
3 (b) A newspaper reported that the unemployment rate was getting worse more slowly. If $U(t)$ is the unemployment rate at time t what does this say about the signs of U' and U'' at the time of the report?



$u' > 0$ POS. (u INCREASING)

$u'' < 0$ NEG. (u CONCAVE DOWN)

(10) A farmer has 4800 feet of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field which has the largest area?



MAXIMIZE : $A = lw$

CONSTRAINT : $l + 2w = 4800$

$l = 4800 - 2w$

$A(w) = (4800-2w)w = 4800w - 2w^2$, $0 \le w \le 2400$

$A'(w) = 4800 - 4w = 4(1200 - w) = 0$

$$\boxed{\begin{array}{l} w = 1200 \\ l = 2400 \end{array}}$$

w	A(w)	
0	0	
1200	2,880,000	(MAX)
2400	0	