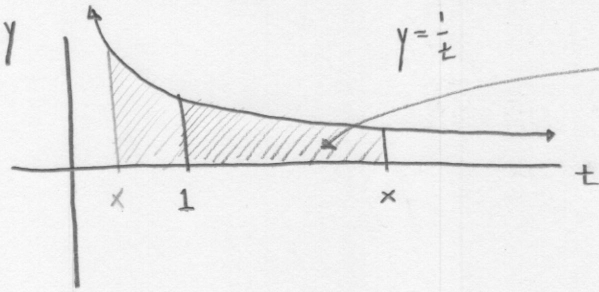
§7.1 THE INTEGRAL DEFINITION OF LOGARITHM

DOMAIN

Def: FOR ALL $x > 0$ DEFINE $\ln x = \displaystyle\int_1^x \frac{1}{t}\,dt$.



$\ln x$ = AREA
- POSITIVE IF $x > 1$
- NEGATIVE IF $x < 1$
- 0 IF $x = 1$

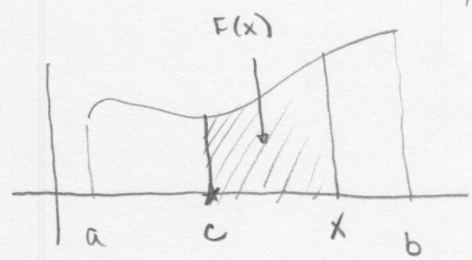
FUNDAMENTAL THM. OF CALC:

LET f BE CONTINUOUS ON $[a, b]$

THEN FOR ANY $c \in [a, b]$ ("EPSILON" $\in$: "BELONGS TO")

WE CAN DEFINE A FUNCTION

$$F(x) = \int_c^x f(t)\,dt$$



(i) $\dfrac{d}{dx} F(x) = f(x)$

(ii) $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$

SINCE $f(x) = \frac{1}{x}$ IS CONTINUOUS ON $(0, \infty)$

FOR ANY $0 < a < b < \infty$, WE CAN DEFINE A LOGARITHM

$$\text{Log}\, x = \int_c^x \frac{1}{t}\,dt, \quad \text{FOR ALL } c > 0.$$

THE NATURAL LOGARITHM HAS $c = 1$.

L DENOTED $\ln x$, $\ln(x)$

NOTE: $\forall c > 0$ ($\forall$: "FOR ALL / FOR ANY")

$\forall x > 0$,

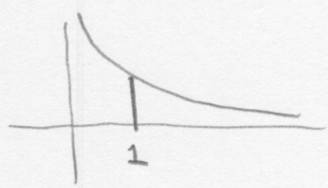
$$\int_c^x \frac{1}{t}\,dt = \int_c^1 \frac{1}{t}\,dt + \int_1^x \frac{1}{t}\,dX$$

$$= -\int_1^c \frac{1}{t}\,dt + \int_1^x \frac{1}{t}\,dt = \underbrace{-\ln c}_{\text{CONSTANT}} + \underbrace{\ln x}_{\text{NATURAL LOGARITHM}}$$

PROPERTIES OF THE NATURAL LOGARITHM

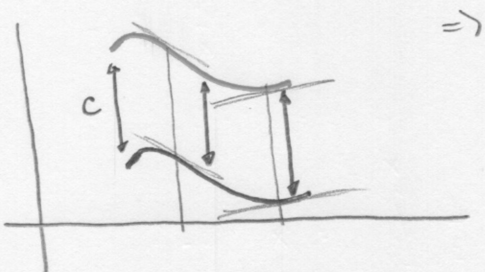
1. $\frac{d}{dx} \ln x = \frac{1}{x}$ (FROM DEFINITION)

2. $\ln(1) = \int_1^1 \frac{1}{t}\,dt = 0$



3. $\frac{d}{dx} \ln(ax) = \frac{1}{ax} \cdot a$ CHAIN RULE $= \frac{1}{x} = \frac{d}{dx} \ln x$

SAME DERIVATIVE



$\Rightarrow$ THEY DIFFER BY A CONSTANT C

$\Rightarrow \ln(ax) = \ln x + C$

set $x = 1$: $\ln(a) = 0 + C$

$(x = b)$ $\therefore$ $\boxed{\ln(ab) = \ln(a) + \ln(b)}$

4. $\frac{d}{dx} \ln(x^a) = \frac{1}{x^a} \cdot a x^{a-1} = a \frac{1}{x} = \frac{d}{dx}[a \ln x]$

SAME DERIVATIVE

$\Rightarrow \ln(x^a) = a \ln x + c$

set $x = 1$: $\ln(1) = 0 = \underbrace{a \ln(1)}_{0} + c = c$

$\Rightarrow c = 0$

set $a = y$ $\quad \therefore \quad \boxed{\ln(x^y) = y \ln x}$

ex. FIND DERIVATIVE

i) $\frac{d}{dx}\left[\ln(x^3 - x^2 + 1)\right]$ $\quad u = x^3 - x^2 + 1$

$$\frac{d}{dx}[\ln u] = \frac{d}{du}[\ln u] \cdot \frac{du}{dx}$$

$$= \frac{1}{u} \cdot (3x^2 - 2x) = \boxed{\frac{3x^2 - 2x}{x^3 - x^2 + 1}} \checkmark$$

i) $\frac{d}{dx}\left[\ln(\ln x)\right] = \frac{1}{\ln x} \cdot \frac{1}{x} = \boxed{\frac{1}{x \ln x}}$

i) $\frac{d}{dx} \frac{\sqrt[3]{7x^2 + 1}}{\sqrt[4]{x - 2}\,\sqrt{x^4 + 1}}$: LET $y = \left(\frac{\sqrt[3]{7x^2 + 1}}{\sqrt[4]{x - 2}\,\sqrt{x^4 + 1}}\right)$

FIND y'.

LOGARITHMIC DIFFERENTIATION :

$$\ln y = \ln\left(\qquad\right) = \ln\left((7x^2+1)^{1/3}\right) - \ln\left((x-2)^{1/4}\cdot(x^4+1)^{1/2}\right)$$

LOG OF BOTH SIDES

$$= \frac{1}{3}\ln(7x^2+1) - \frac{1}{4}\ln(x-2) - \frac{1}{2}\ln(x^4+1)$$

$$\frac{d}{dx}\left[\ln y\right] = \frac{d}{dx}\left[\frac{1}{3}\ln(7x^2+1) - \frac{1}{4}\ln(x-2) - \frac{1}{2}\ln(x^4+1)\right]$$

FIND y'. IMPLICIT DIFFERENTIATION

$$\frac{1}{y}y' = \frac{1}{3}\cdot\frac{1}{7x^2+1}\cdot 14x - \frac{1}{4}\cdot\frac{1}{x-2} - \frac{1}{2}\cdot\frac{1}{x^4+1}\cdot 4x^3$$

$$y' = y\left(\frac{14x}{3(7x^2+1)} - \frac{1}{4(x-2)} - \frac{4x^3}{2(x^4+1)}\right)$$

SUBSTITUTE

ex. Note:

$$\frac{d}{dx}\ln|x| = \begin{cases} \frac{d}{dx}\ln x \text{ IF } x>0 & = \frac{1}{x} \\ \frac{d}{dx}\ln(-x) \text{ IF } x<0 & = \frac{1}{-x}\cdot(-1) = \frac{1}{x}\end{cases}$$

$$\therefore \boxed{\int \frac{1}{x}\,dx = \ln|x| + C}$$

ex.

$$\int \frac{\overset{u}{\ln x}}{x}\,dx \qquad u = \ln x \qquad du = \frac{1}{x}\,dx$$

$$\rightsquigarrow \int u\,du = \frac{1}{2}u^2 + C$$

$$\rightsquigarrow \boxed{\frac{1}{2}(\ln x)^2 + C}$$

ex. $\displaystyle\int \frac{x}{x^2+3}\,dx$

$u = x^2 + 3$

$\frac{du}{dx} = 2x \to du = 2x\,dx$

$\frac{1}{2}\,du = x\,dx$

$$= \int \underbrace{\frac{1}{x^2+3}}_{\frac{1}{u}} \cdot \frac{1}{2} \cdot \underbrace{2x\,dx}_{du} = \int \frac{1}{u}\cdot\frac{1}{2}\,du = \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \frac{1}{2}\ln|u| + C \to \boxed{\frac{1}{2}\ln|x^2+3| + C}$$

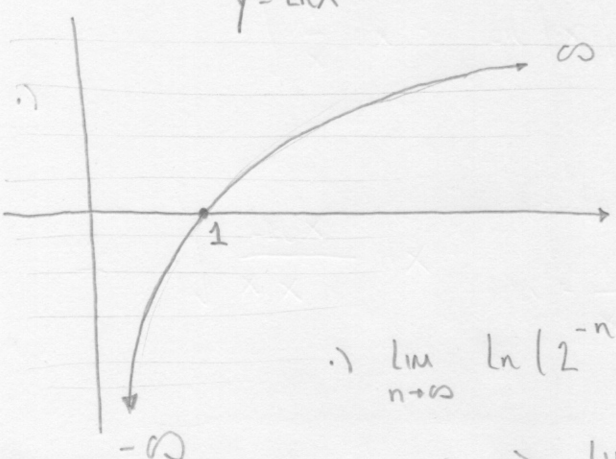
ABS. VAL. SIGNS NOT NECESSARY.

- $\ln x$ IS INCREASING: $\frac{d}{dx}\ln x = \frac{1}{x} > 0$ $\quad(\text{DOM}(\ln): x > 0)$

POSITIVE DERIVATIVE.

- $y = \ln x$ IS CONCAVE DOWN: $\frac{d^2}{dx^2}\ln x = \frac{d}{dx}\frac{1}{x} = \frac{-1}{x^2} < 0$

$y = \ln x$



- $\lim\limits_{n\to\infty} \ln(2^n) = \lim\limits_{n\to\infty} n\underbrace{\ln(2)}_{\text{POS.}} = \infty$

$\Rightarrow \lim\limits_{x\to\infty} \ln(x) = \infty$

- $\lim\limits_{n\to\infty} \ln(2^{-n}) = \lim\limits_{n\to\infty} -n\ln(2) = -\infty$

$\Rightarrow \lim\limits_{x\to 0^+} \ln x = -\infty$

RANGE OF ln: $(-\infty, \infty)$