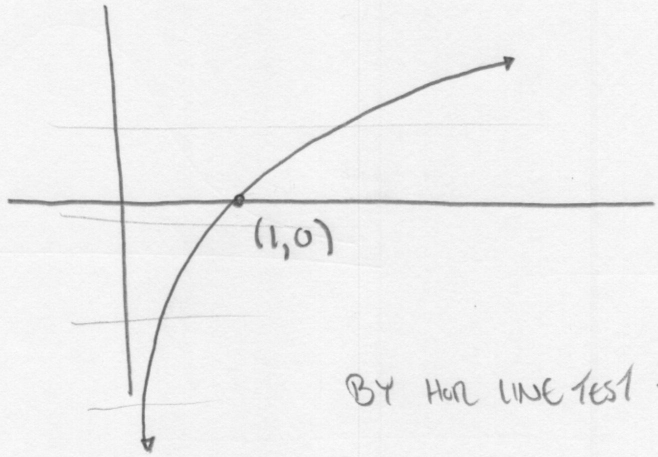
§7.1 Logarithm Cont'd

$y = \ln x$

$\left(= \int_1^x \frac{1}{t}\,dt \right)$



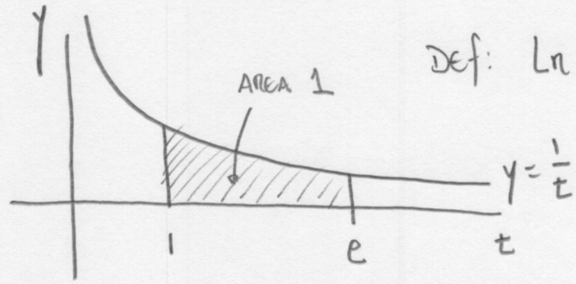
DOMAIN: $(0, \infty)$

RANGE: $(-\infty, \infty)$

BY HOR LINE TEST $\rightarrow$ INVERTIBLE.

LET'S CALL THE INVERSE OF $\ln x$ $\quad E(x)$ $\quad$ [WE WILL SHOW $E(x) = e^x$]

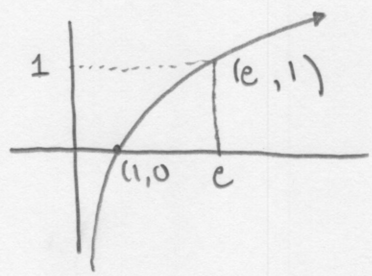
$\ln(x) = \int_1^x \frac{1}{t}\,dt$



Def: $\ln(e) = 1$

≈ 2.718281828

Def: $\ln e = 1$

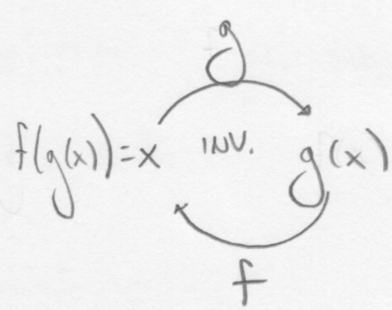


$$\boxed{\begin{aligned} \ln(1) &= 0 \\ \ln(e) &= 1 \end{aligned}}$$

NOTE: FOR ALL $x \in \mathbb{R}$,

$$x = x \cdot 1 = x \cdot \ln e = \ln(e^x)$$

RECALL $x = f(g(x))$ FOR ALL x

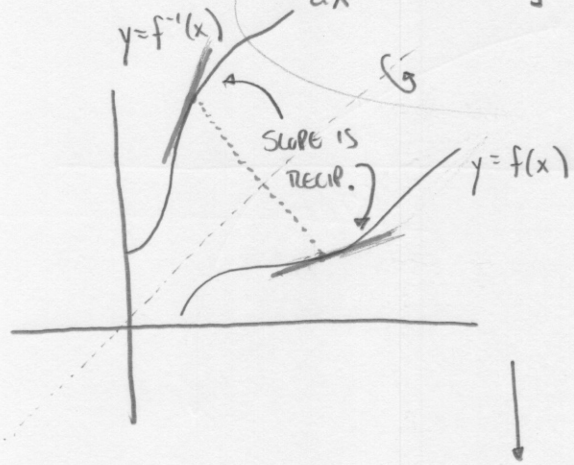


$\therefore$ e^x IS INVERSE OF $\ln x$.

$$\boxed{\frac{d}{dx}\left[e^x\right] = e^x}$$

RECALL INV. FUNC. THM.

$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'\left(\underbrace{f^{-1}(x)}\right)}$$



DIFFERENTIABLE
"SMOOTH"

$$\frac{d}{dx}\left[e^x\right] = \frac{d}{dx}\left[\ln^{-1}(x)\right] = \frac{1}{\frac{1}{\ln^{-1}x}} = \ln^{-1}x = e^x$$

COROLLARY: $\boxed{g(x) = e^{f(x)} \rightarrow g'(x) = e^{f(x)}\, f'(x)}$

COMPOSITION

ex. $\displaystyle\int \underbrace{3\cos x}\; e^{4\sin x}\, dx$

INNER FUNCTION

$u = 4\sin x$

$\frac{3}{4}\ \boxed{du = \cancel{4}\cos x\, dx}\ \frac{3}{\cancel{4}}$

$\frac{3}{4}\,du = \underbrace{3\cos x\, dx}$

$$\int \underbrace{e^{4\sin x}}_{e^u} \cdot \underbrace{3\cos x\, dx}_{\frac{3}{4}du} \rightarrow \frac{3}{4}\int e^u\, du$$

$$\boxed{\int e^x\, dx = e^x + C}$$

$$= \frac{3}{4}e^u + C$$

$$\rightarrow \boxed{\frac{3}{4}e^{4\sin x} + C}$$

NATURAL EXPONENTIAL FUNCTION: e^x BASE $= e \approx 2.718$

GENERAL EXPONENTIAL: a^x, $a > 0$.

$$\boxed{\frac{d}{dx} a^x = a^x \ln a \quad ; \quad \int a^x \, dx = \frac{1}{\ln a} a^x}$$

$$\frac{d}{dx} a^x = \frac{d}{dx} \left(e^{\ln a}\right)^x = \frac{d}{dx} e^{(\ln a) x} = e^{(\ln a) x} \cdot \ln a = a^x \ln a \quad \checkmark$$

CHANGE BASE $\longrightarrow$ $a = e^{\ln a}$ INV. FUNC. PROPERTY

$(a = f(f^{-1}(a)))$

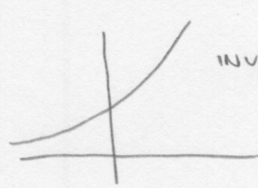
ex. $f(x) = \boxed{\sin(x)}^{\cos(x)} = \boxed{e^{\ln(\sin x)}}^{\cos x}$

$$f(x) = e^{\cos x \ln(\sin x)}$$

$$f'(x) = e^{\cos x \ln(\sin x)} \cdot \left(-\sin x \ln(\sin x) + \cos x \frac{\cos x}{\sin x}\right)$$

GENERAL EXP. FUNC. $f(x) = a^x$, $a > 1$, $a \neq 1$

$= e^{\ln(a) x}$



INVERTIBLE.

THEN CALL $f^{-1}(x) = \log_a x$

LET $y = \log_a x \longrightarrow a^y = x = e^{\ln(a) y}$

$\ln x = \ln\left(e^{\ln(a) y}\right)$

$\ln x = \ln(a) y \underbrace{\ln(e)}_{1}$

$$y = \frac{\ln x}{\ln a}$$

CHANGE OF BASE FORMULA: $\boxed{\log_a x = \dfrac{\ln x}{\ln a}}$

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a} \quad ;$$

INTEGRAL OF $\ln x$ & $\log_a x$ IS COMING SOON!

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ex. $$\int \frac{e^x}{\sqrt{2} - e^x}\, dx$$

$u = \sqrt{2} - e^x$

$\dfrac{du}{dx} = -e^x \Rightarrow du = -e^x\, dx$

$-du = e^x\, dx$

$$\int \underbrace{\frac{1}{\sqrt{2} - e^x}}_{\frac{1}{u}} \cdot \underbrace{e^x\, dx}_{} \rightsquigarrow -\int \frac{1}{u}\, du = -\ln|u| + C$$

$$\rightsquigarrow \boxed{-\ln\left|\sqrt{2} - e^x\right| + C}$$

Note: $$\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + C$$

$$\frac{d}{dx} \ln|f(x)| = \frac{1}{f(x)} \cdot f'(x) \quad \checkmark$$

§7.3 HYPERBOLIC FUNCTIONS

Def:

HYPERBOLIC SINE FUNC.

$$\sinh = \frac{e^x - e^{-x}}{2}$$

"SINCH"

RHYMES W/ PINCH

HYPERBOLIC COSINE FUNC.

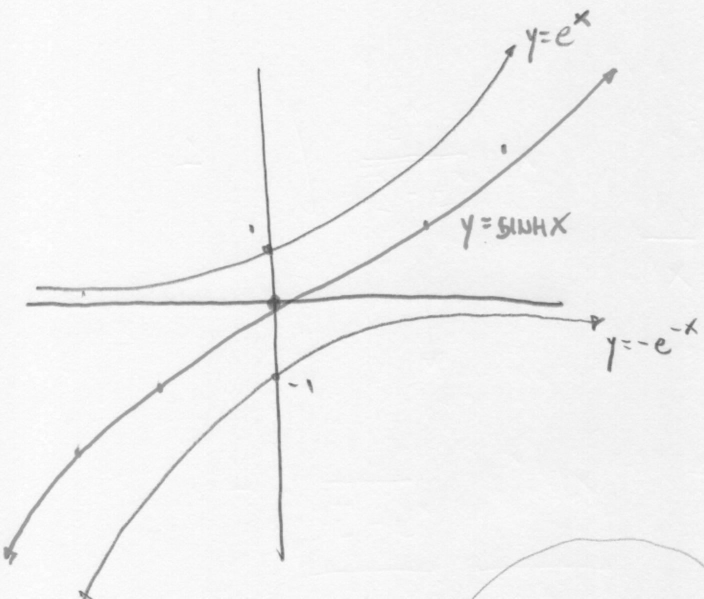
$$\cosh = \frac{e^x + e^{-x}}{2}$$

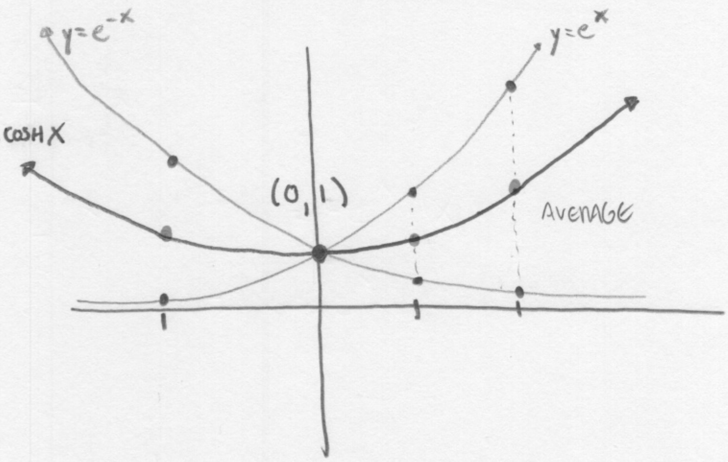
ADD & DIV. BY 2

$\Rightarrow$ AVERAGE

"KOSHE"

LIKE KOSHER SALT





$$\frac{d}{dx} \sinh x = \frac{d}{dx}\left[\frac{e^x - e^{-x}}{2}\right] = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\frac{d}{dx} \cosh x = \frac{d}{dx}\left[\frac{e^x + e^{-x}}{2}\right] = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$\left(\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x} \qquad \operatorname{sech} x = \frac{1}{\cosh x} \qquad \operatorname{csch} x = \frac{1}{\sinh x} \right)$$

SOL'N SET = $\mathbb{R}$

HYPERBOLIC IDENTITIES: $\cosh^2 x - \sinh^2 x = 1$ FOR ALL x.

PROOF: $$\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \dots$$

$$\left(\tfrac{1}{2}\right)^2 \left(e^x + e^{-x}\right)^2 - \left(\tfrac{1}{2}\right)^2 \left(e^x - e^{-x}\right)^2 =$$

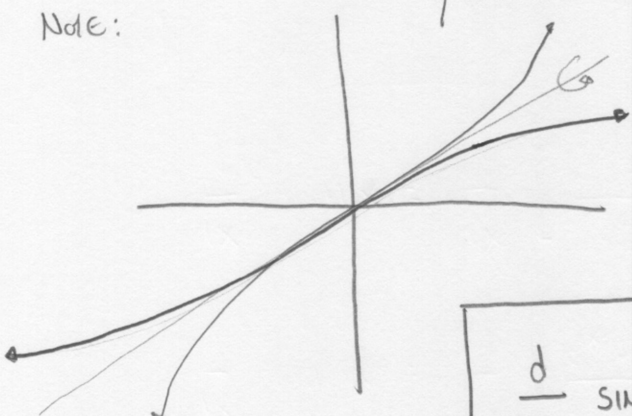
$$\tfrac{1}{4}\left(\cancel{e^{2x}} + 2 + \cancel{e^{-2x}}\right) - \tfrac{1}{4}\left(\cancel{e^{2x}} - 2 + \cancel{e^{-2x}}\right) = \frac{4}{4} = 1 \checkmark$$

THERE ARE OTHER IDENTITIES: ... $\cosh(a \pm b) = \dots$

$\sinh(a \pm b) = \dots$

NOTE:

$y = \sinh x$ IS INVERTIBLE. ($\sinh^{-1} x$)



$\sinh x$ IS DIFF. $\Rightarrow$ $\sinh^{-1} x$ IS DIFF.

BUILT FROM e^x — BUILT FROM $\ln x$

$$\boxed{\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{x^2+1}}}$$

$$\boxed{\int \frac{1}{\sqrt{x^2+1}}\, dx = \sinh^{-1} x + C}$$

LET $y = \sinh^{-1} x$. FIND y'.

$\sinh y = x \rightarrow \cosh y \cdot y' = 1 \rightarrow y' = \dfrac{1}{\cosh y}$

$\cosh^2 x - \sinh^2 x = 1$

$\cosh x = \pm\sqrt{1 + \sinh^2 x}$

$$y' = \frac{1}{\pm\sqrt{1+\sinh^2 y}} = \frac{1}{\pm\sqrt{1+x^2}}$$

POS $\sqrt{\ }$

§8.1 Using Basic Integration Formulas

0. u-sub,

1. Completing the square

u-sub.

ex. $\int \frac{1}{\sqrt{x^2-6x+5}}\, dx$ ← complete the sq.

$$\boxed{\underline{x^2-6x+5} = \underbrace{(x-3)^2}_{} + \underbrace{-4}_{\text{const.}}}$$

$$\overbrace{x^2-6x+9}^{} - 4$$

$$= \int \frac{1}{\sqrt{(x-3)^2-4}}\, dx$$

Let $u = x-3$

$du = dx$

$$= \int \frac{1}{\sqrt{u^2-4}}\, du = \int \frac{1}{\sqrt{4}\sqrt{\frac{u^2}{4}-1}}\, du$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{(\frac{u}{2})^2-1}}\, du$$

Let $v = \frac{u}{2}$

$dv = \frac{1}{2}\, du$

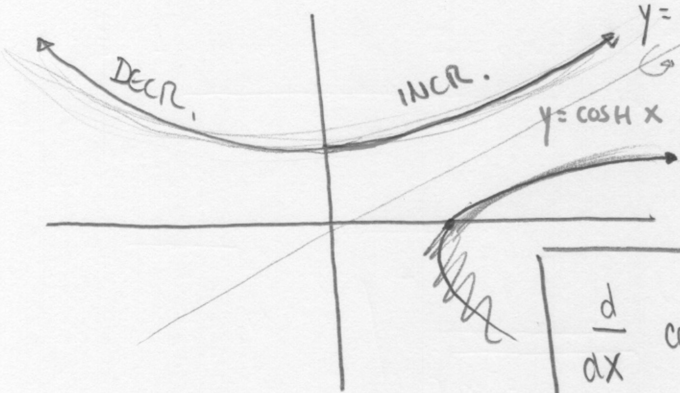
$2\,dv = du$

$$\to = \cancel{\frac{1}{2}}\cdot\cancel{2}\ \boxed{\int \frac{1}{\sqrt{v^2-1}}\, dv = \cosh^{-1} v + C}$$

$$\boxed{\int \frac{1}{\sqrt{x^2+1}}\, dx = \sinh^{-1} x + C}$$

$\to \cosh^{-1}\frac{u}{2} + C$

$\to \boxed{\cosh^{-1}\frac{x-3}{2} + C}$



$y = \cosh x$, $x \geq 0$, $y \geq 1$ INVERTIBLE

DEFINE ITS INVERSE TO BE $\cosh^{-1} x$

$$\boxed{\frac{d}{dx} \cosh^{-1} x = \frac{1}{\sqrt{x^2 - 1}}} \quad , \; x \geq 1$$

LET $y = \cosh^{-1} x$. FIND y'. $\left(\begin{array}{l} x \geq 1 \\ y' \text{ + POSITIVE} \end{array} \right)$

$\cosh y = x$ IMPLICIT DIFF. $\cosh^2 y - \sinh^2 y = 1$

$$\sinh y \cdot y' = 1 \rightarrow y' = \frac{1}{\sinh y} = \frac{1}{\pm\sqrt{\cosh^2 y - 1}}$$

$$\boxed{y' = \frac{\pm 1}{\sqrt{x^2 - 1}}}$$

~~+ IF $x \geq 0$~~

~~− IF $x < 0$~~