ex. $\displaystyle\int_0^{\pi/4} 3^{\boxed{\cos(2t)}} \boxed{\sin(2t)\,dt}$ -

$y = a^x = e^{(\ln a)x}$

$y' = e^{(\ln a)x} \ln a = a^x \ln a$

$$\boxed{\frac{d}{dx} a^x = a^x \ln a \quad \Big\} \quad \int a^x\,dx = \frac{1}{\ln a} a^x + C}$$

$\boxed{u = \cos(2t)}$

$du = -2\sin(2t)\,dt$

$-\frac{1}{2}\,du = \sin(2t)\,dt$

CHANGE VARIABLE $\Rightarrow$ CHANGE BOUNDS

$t = \pi/4 \longrightarrow u = \cos(2 \cdot \frac{\pi}{4}) = 0$

$t = 0 \longrightarrow u = \cos(2 \cdot 0) = 1$

$$-\frac{1}{2}\int_1^0 3^u\,du$$

SWITCH BOUNDS + NEGATE

$$\frac{1}{2}\int_0^1 3^u\,du = \frac{1}{2}\left[\frac{1}{\ln 3} 3^u\right]_0^1$$

$$= \frac{1}{2\ln 3}\left(3^1 - 3^0\right) = \frac{\cancel{2}}{\cancel{2}\ln 3} = \boxed{\frac{1}{\ln 3}}$$

§8.1 USING BASIC INTEGRATION FORMULAS

0. u-SUBSTITUTION
1. COMPLETE THE SQ.
2. TRIG IDENTITIES

PYTHAGOREAN IDENTITY

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 = \sec^2\theta - \tan^2\theta \quad *$$

ex. $\displaystyle\int \frac{1}{\sec\theta + \tan\theta}\, d\theta$

$$= \int \frac{(\sec\theta - \tan\theta)}{(\sec\theta + \tan\theta)(\sec\theta - \tan\theta)}\, d\theta$$

$(a+b)(a-b)$

$a^2 - b^2$

$$= \int \frac{\sec\theta - \tan\theta}{\sec^2\theta - \tan^2\theta\ *}\, d\theta = \int \sec\theta - \tan\theta\ d\theta$$

$$= \underset{①}{\int \sec\theta\, d\theta} - \underset{②}{\int \tan\theta\, d\theta}$$

② $\displaystyle\int \tan\theta\, d\theta$

$$= \int \frac{\sin\theta}{\cos\theta}\, d\theta$$

$u = \cos\theta$

$du = -\sin\theta\, d\theta$

$-du = \sin\theta\, d\theta$

$$\rightsquigarrow \int -\frac{1}{u}\, du$$

$$= -\ln|u| = -\ln|\cos\theta| + C$$

① $\displaystyle\int \sec\theta\, d\theta = \int \sec\theta \cdot \frac{\tan\theta + \sec\theta}{\tan\theta + \sec\theta}\, d\theta$

$$= \int \frac{\sec\theta\tan\theta + \sec^2\theta}{\tan\theta + \sec\theta}\, d\theta$$

LET $u = \tan\theta + \sec\theta$

$du = (\sec\theta\tan\theta + \sec^2\theta)\, d\theta$

$$\rightsquigarrow \int \frac{1}{u}\, du = \ln|u| + C \rightsquigarrow \underset{①}{\ln|\tan\theta + \sec\theta| + C}$$

$$② \ldots \ -\ln|\cos\theta| + c = \ln\left(|\cos\theta|^{-1}\right) + c$$

$$= \ln|\sec\theta| + c$$

$$\therefore \ ① - ② = \ln|\tan\theta + \sec\theta| - \ln|\sec\theta| + c$$

3. MULTIPLY TOP & BOTTOM OF FRACTION BY <u>RADICAL CONJUGATES</u>

EXPRESSION	CONJUGATE	"COUSIN"	TRIG IDENTITIES!
$a + b$	$a - b$	MULTIPLY: $(a+b)(a-b) = a^2 - b^2$	
$a + ib$	$a - ib$	"COMPLEX CONJUGATE"	
$a + \sqrt{b}$	$a - \sqrt{b}$	"RADICAL CONJUGATE"	

$$(a+\sqrt{b})(a-\sqrt{b}) = a^2 - \cancel{a\sqrt{b}} + \cancel{a\sqrt{b}} - b$$

$$= a^2 - b \quad \text{(NO RADICAL!)}$$

<u>ex.</u>

$$\int \sqrt{\boxed{1-\cos\theta}}\, d\theta = \int \sqrt{\frac{(1-\cos\theta)(1+\cos\theta)}{(1+\cos\theta)}}\, d\theta$$

$$= \int \sqrt{\frac{1-\cos^2\theta}{1+\cos\theta}}\, d\theta = \int \sqrt{\frac{\sin^2\theta}{1+\cos\theta}}\, d\theta$$

$$= \int \frac{\sin\theta}{\sqrt{1+\cos\theta}}\, d\theta \qquad \text{LET } u = 1+\cos\theta$$

$$-du = \sin\theta\, d\theta$$

$$\rightsquigarrow \quad -\int \frac{1}{\sqrt{u}}\,du = -\int u^{-1/2}\,du = -2u^{1/2} + C$$

$$\rightsquigarrow \quad \boxed{2\left(1+\cos\theta\right)^{1/2} + C}$$

4. <u>POLYNOMIAL LONG DIVISION.</u>

"IMPROPER"

DEGREE OF NUM. BIGGER THAN ($\geq$) DEGREE OF DENOM.

$\frac{3}{7}$ "PROPER" $3 < 7$

ex. $$\int \frac{8x^2 + 10x + 3}{4x+5}\,dx$$

$$\begin{array}{r|l} & 2x \\ \hline 4x+5 & 8x^2 + 10x + 3 \\ & -(8x^2 + 10x) \\ \hline & 0 \quad\; 0 \quad\; 3 \end{array}$$

3 REMAINDER

$$\begin{array}{r|l} & 6 \\ \hline 7 & 43 \\ & -42 \\ \hline & 1 \end{array}$$

1 REMAINDER

$\frac{43}{7} = 6 + \frac{1}{7}$

$\frac{43}{7}$ "IMPROPER" $\rightarrow 6\frac{1}{7}$

$$\int 2x + \frac{3}{4x+5}\,dx$$

$$= x^2 + C + \int \frac{3}{4x+5}\,dx$$

LET $u = 4x+5$

$du = 4\,dx$

$\frac{1}{4}\,du = dx$

$$= x^2 + C + \frac{3}{4}\int \frac{1}{u}\,du = x^2 + \frac{3}{4}\ln|u| + C$$

$$\rightsquigarrow \quad \boxed{x^2 + \frac{3}{4}\ln|4x+5| + C}$$

SUMMARY: $\displaystyle\int \frac{\text{DEGREE } n \text{ POLYNOMIAL}}{\text{DEGREE } m \text{ POLYNOMIAL}}\,dx$

USE POLYNOMIAL LONG DIVISION WHEN $n \geq m$.

(DEGREE = HIGHEST EXPONENT)

ex. $\displaystyle\underbrace{\int \frac{x^4}{x^2+1}\,dx}_{\text{LONG DIV.}}$ vs. $\displaystyle\underbrace{\int \frac{x^2+1}{x^4}\,dx}_{\text{NOT LONG DIV.}}$

LONG DIV.

$$
\begin{array}{r|l}
 & x^2 - 1 \\
\hline
x^2+1 & x^4 \\
 & -(x^4 + x^2) \\
\hline
 & -x^2 \\
 & -(-x^2 - 1) \\
\hline
 & 1
\end{array}
$$

1 REMAINDER

$$= \int x^2 - 1 + \frac{1}{x^2+1}\,dx$$

$$= \boxed{\frac{1}{3}x^3 - x + \tan^{-1}x + C}$$

NOT LONG DIV.

$$\int \frac{x^2}{x^4} + \frac{1}{x^4}\,dx = \int x^{-2} + x^{-4}\,dx$$

$$= \boxed{-x^{-1} - \frac{1}{3}x^{-3} + C}$$

$$\boxed{\frac{d}{dx}\tan^{-1}x = \frac{1}{x^2+1}} \Rightarrow \int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C$$

LET $y = \tan^{-1} x$. FIND y'.

SAME METHOD SHOWS

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$

$\tan y = x$. IMPLICIT DIFFERENTIATION.

$$\sec^2 y \cdot y' = 1 \Rightarrow y' = \frac{1}{\sec^2 y} = \frac{1}{(\tan y)^2 + 1} = \frac{1}{x^2+1} \checkmark$$

$$\left(\tan^2 y + 1 = \sec^2 y\right)$$

$$\boxed{\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C, \quad a>0 \qquad \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C, \quad a>0}$$

FORCE FACTORING:

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \int \frac{1}{\sqrt{a^2\left(1-\frac{x^2}{a^2}\right)}}\,dx = \frac{1}{\sqrt{a^2}}\int \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\,dx \qquad u = \frac{x}{a}$$

$$du = \frac{1}{a}\,dx$$

$$= \cancel{\frac{1}{a}}\,\cancel{a}\int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}u + C \qquad a\,du = dx$$

$$\rightsquigarrow \sin^{-1}\left(\frac{x}{a}\right) + C \checkmark$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a^2}\int \frac{1}{\frac{x^2}{a^2}+1}\,dx = \frac{1}{a^2}\int \frac{1}{\left(\frac{x}{a}\right)^2+1}\,dx$$

$$u = \frac{x}{a} \ldots$$

5. SPLITTING UP A FRACTION

ex. $$\int \frac{x + 2\sqrt{x-1}}{2x\sqrt{x-1}}\,dx$$

$$= \int \frac{\cancel{x}}{2\cancel{x}\sqrt{x-1}}\,dx + \int \frac{\cancel{2\sqrt{x-1}}}{\cancel{2}x\cancel{\sqrt{x-1}}}\,dx$$

$$= \frac{1}{2}\int (x-1)^{-1/2}\,dx + \int \frac{1}{x}\,dx$$

$$\begin{pmatrix} u = x-1 \\ du = dx \end{pmatrix}$$

$$= \boxed{(x-1)^{1/2} + \ln|x| + c}$$

FORCING A MULTIPLE OF DENOM. TO APPEAR IN NUM.

ex. $$\int \frac{1}{1+e^x}\,dx = \int \frac{(1+e^x) - e^x}{(1+e^x)}\,dx$$

$$= \int \underbrace{\frac{1+e^x}{1+e^x}}_{1} - \frac{e^x}{1+e^x}\,dx$$

$$= x - \int \frac{e^x}{1+e^x}\,dx$$

LET $u = 1+e^x$

$du = e^x\,dx$

$$= x - \int \frac{1}{u}\,du = x - \ln|u| + c \rightsquigarrow \boxed{x - \ln|1+e^x| + c}$$

6. "ADVANCED" u- SUBSTITUTION

*

ex. $\int x^7 \sqrt{\underbrace{x^4+1}} \, dx$ Let $u = x^4 + 1 \longrightarrow u - 1 = x^4$

$du = 4x^3 \, dx$

$\frac{1}{4} du = x^3 \, dx$

$$\int \underbrace{x^4}_{u-1} \underbrace{\sqrt{x^4+1}}_{\sqrt{u}} \underbrace{x^3 \, dx}_{\frac{1}{4} du} \rightsquigarrow \frac{1}{4} \int (u-1) u^{1/2} \, du$$

(ALL u's)

$$= \frac{1}{4} \int u^{3/2} - u^{1/2} \, du = \frac{1}{4}\left(\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right) + C$$

$$\rightsquigarrow \frac{1}{4}\left[\frac{2}{5}(x^4+1)^{5/2} - \frac{2}{3}(x^4+1)^{3/2} \right] + C$$

---

ex. $\int \frac{\cos x}{a + \cos x} \, dx$ ~~$u = a + \cos x$~~ ~~$du = -\sin x \, dx$~~ ?

$$= \int \frac{\cos x + (a) - (a)}{a + \cos x} \, dx$$

HW: 7.1, 7.3, 8.1

$$= \int \underbrace{\frac{\cos x + a}{a + \cos x}}_{1} - \underbrace{\frac{a}{a + \cos x}}_{?} \, dx$$