EVALUATE

$$\int \frac{6}{\sqrt{25-36x^2}}\,dx$$

$$u = 6x$$

$$du = 6\,dx$$

$$\int \frac{1}{\sqrt{25-u^2}}\,du$$

$$= \sin^{-1}\frac{u}{5} + C$$

$$\sin^{-1}\frac{6x}{5} + C$$

SIMPLIFY

$$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

$$5\sinh(4\ln x)$$

$$\frac{5}{2}\left(e^{4\ln x} - e^{-4\ln x}\right)$$

$$\frac{5}{2}\left[\left(\cancel{e}^{\cancel{\ln} x}\right)^4 - \left(\cancel{e}^{\cancel{\ln} x}\right)^{-4}\right]$$

$$\frac{5}{2}\left(x^4 - x^{-4}\right)$$

$$\left(\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C\right)$$

§8.2 INTEGRATION BY PARTS

(PRODUCT RULE FOR INTEGRATION)

LET $u(x)$ & $v(x)$ BE DIFFERENTIABLE FUNCTIONS OF X.

PRODUCT RULE: $\frac{d}{dx}\left[u(x)v(x)\right] = u'(x)v(x) + u(x)v'(x)$

F.T.C. $\Rightarrow$ $u(x)v(x) = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx$

REARRANGE $\int \underbrace{u(x)}_{u}\,\underbrace{v'(x)\,dx}_{dv} = u(x)v(x) - \int \underbrace{v(x)}_{v}\,\underbrace{u'(x)\,dx}_{du}$

$$\left(\begin{array}{ll} \frac{dv}{dx} = v'(x) & \frac{du}{dx} = u'(x) \\ v'(x)\,dx = dv & du = u'(x)\,dx \end{array} \right)$$

i.e. $\boxed{\int u\,dv = uv - \int v\,du}$

THE POINT: ONE INTEGRAL: $\int u(x)\,v'(x)\,dx$

ANTI-DERIV. DERIV.

IS REPLACED WITH ANOTHER $\int v(x)\,u'(x)\,dx$

DIFFERENT PRODUCT OF FUNCTIONS.

(HOPEFULLY SIMPLER TO INTEGRATE!)

ex. $\int x^3 \ln x \, dx$

U-SUB FOR FUNCTION COMPOSITION

i.e. $f(g(x)) \sim f(u)$

INTEGRATION BY PARTS FOR PRODUCTS OF FUNC.

DERIV. $u = \ln x$, $du = \frac{1}{x}\,dx$

ANTI-DERIV $v = \frac{1}{4}x^4$, $dv = x^3\,dx$ $\left(\frac{dv}{dx} = x^3 \right)$

$$\int u\,dv = uv - \int v\,du$$

$$= (\ln x)\left(\frac{1}{4}x^4\right) - \int \left(\frac{1}{4}x^4\right)\frac{1}{x}\,dx$$

$$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3\,dx$$

$$= \boxed{\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C}$$

HOW TO DECIDE WHAT TO SET EQUAL TO u & dv ?

$u = \bigcirc$ $\qquad$ $v =$ ANTI-DERV.

$du =$ DERIV. $\qquad$ $dv = \bigcirc$

TIP: CHOOSE u TO BE A FUNCTION THAT CAN BE DIFFERENTIATED TO GET A SIMPLER FUNCTION.

CHECKLIST FOR CHOOSING u:

1. Logarithm.
2. Inverse functions.
3. Algebraic functions. — Polynomials, $\sqrt[n]{}$,
4. Trigonometric functions.
5. Exponential functions.

① $u = \sin x$ $\quad v = \frac{1}{3}x^3$

$du = \cos x \, dx$ $\quad dv = x^2 dx$

ex. $\int x^2 \sin x \, dx$

② $u = x^2$ $\quad v = -\cos x$

$du = 2x\,dx$ $\quad dv = \sin x \, dx$

① $= uv - \int v\,du = \boxed{\frac{1}{3}x^3 \sin x - \int \frac{1}{3} x^3 \cos x \, dx}$

② $= uv - \int v\,du = -x^2 \cos x + 2 \int \underbrace{\overset{u}{x} \overbrace{\cos x \, dx}^{dv}}$

$u = x$ $\quad v = \sin x$

$du = dx$ $\quad dv = \cos x \, dx$

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2 \left[uv - \int v\,du \right]$$

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2 \left(x \sin x - \int \sin x \, dx \right)$$

$$= \boxed{-x^2 \cos x + 2 \left(x \sin x + \cos x \right) + C}$$

ex. $\int e^x \sin x \, dx$

$u = \sin x \qquad v = e^x$

$du = \cos x \, dx \qquad dv = e^x \, dx$

$\int u\,dv = uv - \int v\,du = e^x \sin x - \int e^x \cos x \, dx$

$u = \cos x \qquad v = e^x$

$du = -\sin x \, dx \qquad dv = e^x \, dx$

$$\int e^x \sin x \, dx = e^x \sin x - \Big[\underbrace{e^x \cos x}_{uv} \underbrace{+ \int e^x \sin x \, dx}_{- \int v \, du} \Big]$$

$$\int e^x \sin x \, dx = e^x \sin x - e^x \cos x - \cancel{\int e^x \sin x \, dx}$$

SOLVE FOR $\int e^x \sin x \, dx$

$$+ \int e^x \sin x \, dx \qquad\qquad + \cancel{\int e^x \sin x \, dx}$$

$$2 \int e^x \sin x \, dx = e^x (\sin x - \cos x)$$

$$\int e^x \sin x \, dx = \boxed{\tfrac{1}{2} e^x (\sin x - \cos x) + C}$$

ex. $\int \ln x \, dx$

$u = \ln x \qquad v = x$

$du = \frac{1}{x} dx \qquad dv = 1 \, dx$

$$\int u \, dv = uv - \int v \, du = x \ln x - \underbrace{\int \cancel{x} \cdot \frac{1}{\cancel{x}} \, dx}_{\int 1 \, dx}$$

$$\boxed{\int \ln x \, dx = x \ln x - x + C}$$

$$\frac{d}{dx}\left[x \ln x - x \right] = \ln x \checkmark$$

FOR DEFINITE INTEGRALS

$$F(x) = \int_a^x f(x) \, dx$$

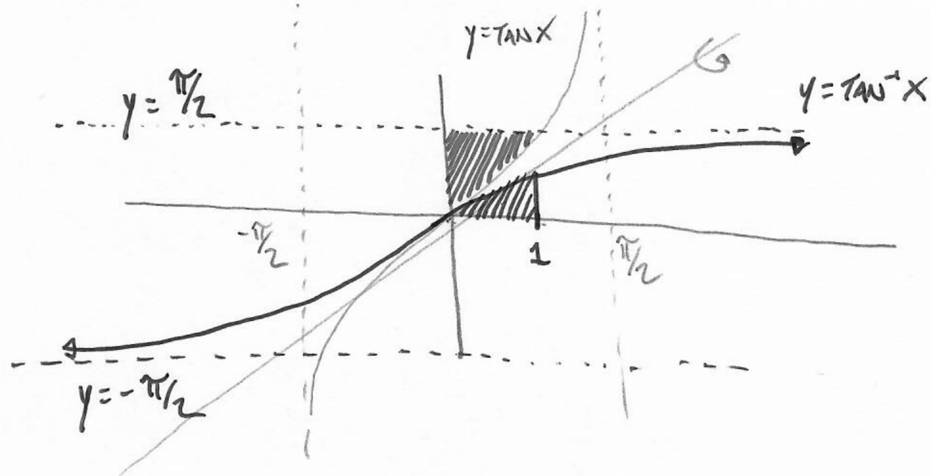
F.T.C. PART II (EVALUATION THM)

(i) $F'(x) = f(x)$

(ii) $\int_a^b f(x) \, dx = F(b) - F(a)$

$$\boxed{\int_a^b u \, dv = uv \Big|_a^b - \int_a^b v \, du}$$

ex. $\displaystyle\int_0^1 \tan^{-1} x \, dx$



L
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$u = \tan^{-1} x \qquad v = x$

$du = \dfrac{1}{1+x^2} dx \qquad dv = 1\, dx$

$u = 1 + x^2$

$du = 2x\, dx$

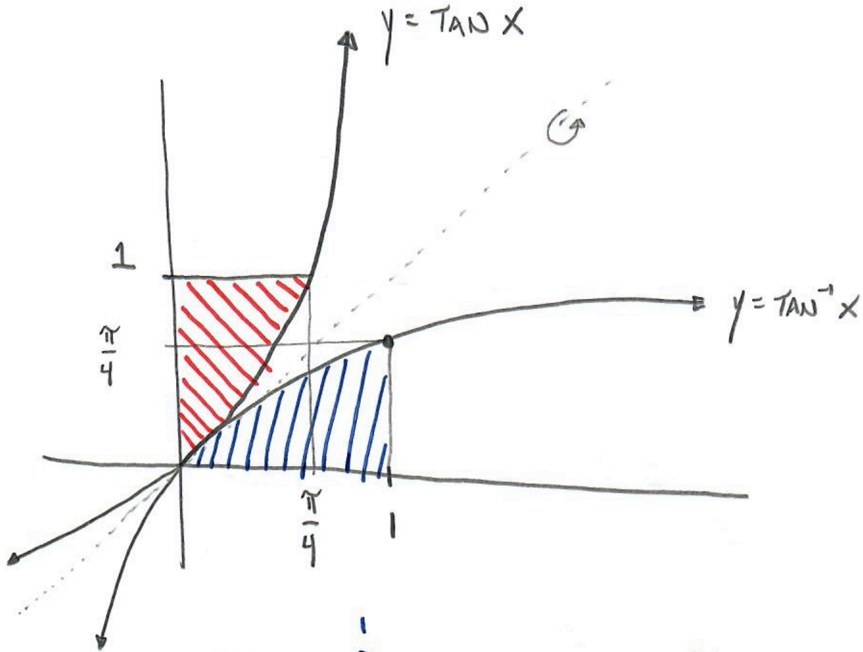
$\frac{1}{2} du = x\, dx$

$$\int u\, dv = uv - \int v\, du = x \tan^{-1} x \Big|_0^1 - \underbrace{\int_0^1 \frac{x}{1+x^2} dx}$$

$$\rightarrow \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln |u|$$

$$\int_0^1 \tan^{-1} x = x \tan^{-1} x \Big|_0^1 - \frac{1}{2} \ln |1 + x^2| \Big|_0^1$$

$$= \tan^{-1}(1) - \frac{1}{2} \ln(2) = \boxed{\frac{\pi}{4} - \frac{1}{2} \ln(2)}$$



$$\boxed{\text{▨}} = \int_0^1 \tan^{-1} x \, dx = \frac{\pi}{4} - \frac{1}{2} \ln 2$$

$$\boxed{\text{▧}} = \frac{\pi}{4} - \int_0^{\pi/4} \tan x \, dx$$

$$= \frac{\pi}{4} - \ln|\sec x| \Big|_0^{\pi/4}$$

$$= \frac{\pi}{4} - \left(\ln\left(\sqrt{2}\right) - 0 \right)$$

✓ (SAME)

$\overset{w}{\cancel{u}}$ - SUB BEFORE INT. BY PARTS.

ex. $\displaystyle\int_0^1 x^5 e^{x^3}\,dx$

LET $w = x^3$

~~LI~~ATE

$dw = 3x^2\,dx$

$\frac{1}{3}\,dw = x^2\,dx$

$$\int_0^1 x^3 x^2 e^{x^3}\,dx = \frac{1}{3}\int_0^1 w e^{w}\,dw$$

$\frac{1}{3}\int_0^1 u\,dv$

$u = w \qquad v = e^w$

$du = dw \qquad dv = e^w\,dw$

$$= \frac{1}{3}\left[uv\Big|_0^1 - \int_0^1 v\,du \right] = \frac{1}{3}\left(we^{w}\Big|_0^1 - \underbrace{\int_0^1 e^{w}\,dw}_{e^w\big|_0^1} \right)$$

$$= \frac{1}{3}\left(1e^{1} - 0e^{0} - \left(e^1 - e^0\right) \right)$$

$$= \frac{1}{3}\left(e - e + 1 \right) = \boxed{\frac{1}{3}}$$

REDUCTION FORMULA FOR SIN & COS

$$\int \sin^n x \, dx = \int \sin^{n-1} x \sin x \, dx$$

$$u = \sin^{n-1} x \qquad v = -\cos x$$

$$du = (n-1)\sin^{n-2} x \cos x \, dx \qquad dv = \sin x \, dx$$

$$\int u\,dv = uv - \int v\,du = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \underbrace{\cos^2 x}_{(1-\sin^2 x)} \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\left[\int \sin^{n-2} x \, dx - \int \sin^n x \, dx\right]$$

$$1\int \sin^n x \, dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \, dx - (n-1)\int \sin^n x \, dx$$

$$+(n-1)\int \sin^n x \, dx \qquad\qquad +(n-1)\int \sin^n x \, dx$$

$$n\int \sin^n x \, dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \, dx$$

REDUCED!
THIS FORMULA CAN BE REPEATED!

$$\boxed{\int \sin^{n} x \, dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x \, dx}$$

SIMILARLY,

$$\boxed{\int \cos^n x \, dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x \, dx}$$

ex. $\int \sin^3 x \, dx$ REDUCTION FORMULA FOR SIN WITH $n = 3$.

$$= -\frac{1}{3}\sin^2 x \cos x + \frac{2}{3}\int \sin^1 x \, dx$$

$$= \boxed{-\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x + C}$$

§8.3 TRIGONOMETRIC INTEGRALS

ALL SIN'S

1 COS

ex. $\int \underbrace{\sin^3 x}_{u^3} \underbrace{\cos x \, dx}_{du}$ LET $u = \sin x$

$du = \cos x \, dx$

$$\rightsquigarrow \int u^3 \, du = \frac{1}{4}u^4 + C \rightsquigarrow \boxed{\frac{1}{4}\sin^4 x + C}$$

ex. $\int \cos^8 x \sin^3 x \, dx$ RECALL: $\sin^2 x + \cos^2 x = 1$

$\sin^2 x = 1 - \cos^2 x$

$$= \int \cos^8 x \sin^2 x \sin x \, dx$$

LET $u = \cos x$

$$= \int \cos^8 x \left(1 - \cos^2 x\right) \sin x \, dx$$

$-du = \sin x \, dx$

$$\rightsquigarrow -\int \underbrace{u^8\left(1 - u^2\right)}_{u^8 - u^{10}} du = -\frac{1}{9}u^9 + \frac{1}{11}u^{11} + C$$

$$= \boxed{-\frac{1}{9}\cos^9 x + \frac{1}{11}\cos^{11} x + C}$$

IN SUMMARY : $\boxed{\int \sin^m x \cos^n x \, dx}$

- IF m IS ODD : $\int (\text{ALL } \cos) \sin x \, dx$

$\sin^2 x = 1 - \cos^2 x$

- IF n IS ODD : $\int (\text{ALL } \sin) \cos x \, dx$

- IF m, n BOTH EVEN : TRIG IDENTITIES

$$* \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

HW $\boxed{7.1, 7.3, 8.1, 8.2}$ ← QUIZ 1 TUESDAY

(8.3 HEAD START)