QUIZ #2 TOMORROW 8:35 ON PEARSON (30 MIN)

CLASS RESUMES: 9:10 AM.

8.3, 8.4, 8.5, 8.7

$x \to \infty \to u \to \infty$

$\underbrace{x = 0}$ $\to u = 0$

8.8 $\int_0^\infty \frac{\boxed{4\,dx}}{(9+x)\boxed{\sqrt{x}}}$

LET $u = \sqrt{x} \to u^2 = x$

$du = \frac{1}{2\sqrt{x}}dx \to 2\,du = \frac{1}{\sqrt{x}}dx$

$\downarrow$

$8\,du = \frac{4}{\sqrt{x}}dx$

$8\int_0^\infty \frac{1}{9+u^2}\,du$

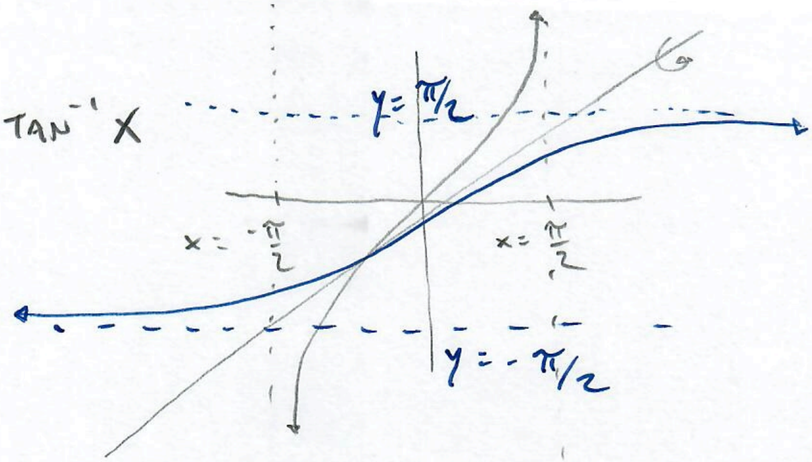
$$\left(\int \frac{1}{u^2+a^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C\right)$$

$$= 8 \lim_{t\to\infty} \int_0^t \frac{1}{9+u^2}\,du$$

$$= 8 \lim_{t\to\infty} \frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right)\Bigg|_0^t$$

$$= \boxed{\frac{8}{3} \lim_{t\to\infty} \tan^{-1}\left(\frac{t}{3}\right)} - \underbrace{\tan^{-1}\left(\frac{0}{3}\right)}_{0}$$

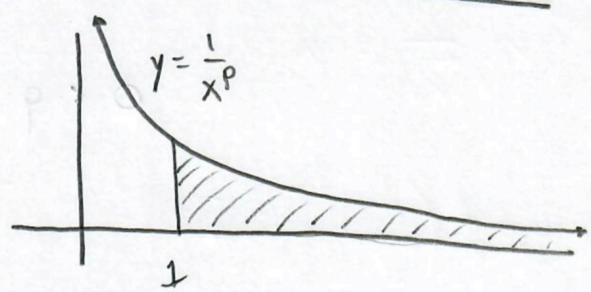
$\lim\limits_{x \to \infty} \tan^{-1} x$



$= \frac{8}{3} \cdot \frac{\pi}{2} = \boxed{\frac{4\pi}{3}}$

P-TEST FOR IMPROPER INTEGRALS OF TYPE I (AREAS OF REGIONS WITH ∞ WIDTH)

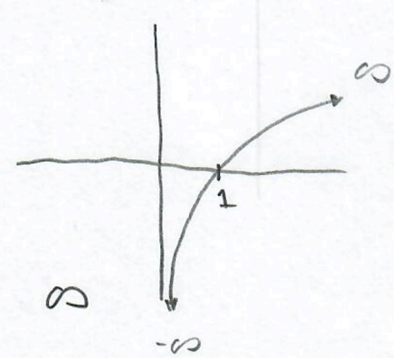
$\int_1^\infty \frac{1}{x^p}\,dx$ "P FOR POWER",



$p > 0$

IF $p = 1$: $\int_1^\infty \frac{1}{x}\,dx = \lim\limits_{t \to \infty} \int_1^t \frac{1}{x}\,dx$

$= \lim\limits_{t \to \infty} \left(\ln t - \underbrace{\ln 1}_{0} \right) = \infty$



$= \infty$

$$\frac{1}{x^p} = x^{-p} \xrightarrow{\text{INT}} \frac{1}{-p+1} x^{-p+1}$$

NOW, ASSUME $p \neq 1$

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}\,dx = \lim_{t\to\infty} \frac{1}{1-p} x^{1-p} \Big|_1^t$$

$$= \lim_{t\to\infty} \frac{1}{1-p}\left(t^{1-p} - 1\right) = \begin{cases} (\infty & \text{IF } p = 1) \\ \infty & \text{IF } p < 1 \\ \dfrac{-1}{1-p} & \text{IF } p > 1 \end{cases}$$

$$\text{IF } 1-p > 0 \quad \lim_{t\to\infty} t^{1-p} = \infty$$

$$\text{IF } 1-p < 0 \quad \lim_{t\to\infty} t^{1-p} = 0$$

$$\therefore \boxed{\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \text{CONVERGES TO } \dfrac{1}{p-1} & \text{IF } p > 1. \\ \text{DIVERGES (TO } \infty\text{)} & \text{IF } p \leq 1. \end{cases}}$$

e.g. $\displaystyle\int_1^\infty \frac{1}{x\sqrt{x}}\,dx \quad \frac{1}{x^{3/2}}.$ $\qquad \displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ $\qquad$ DIVERGES OR CONVERGE ?

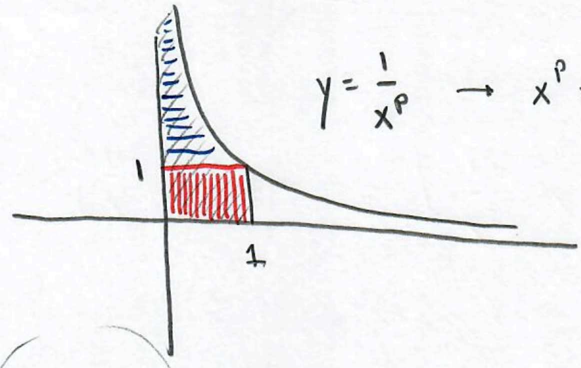
CONVERGES. $\qquad$ DIVERGES

$p = \frac{3}{2}$ $\qquad$ $p = \frac{1}{2}$

CONVERGES TO $\frac{1}{p-1} = 2$

p-TEST FOR IMP. INT. OF TYPE II : (REGIONS WITH ∞ HEIGHT)

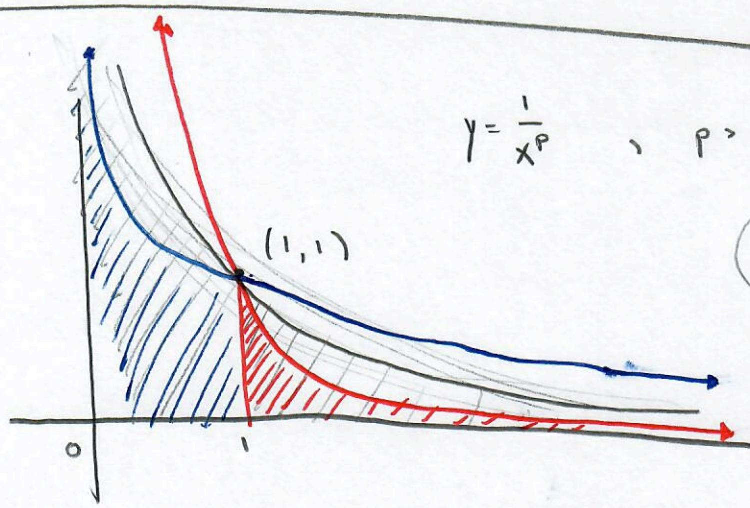
$$\int_0^1 \frac{1}{x^p}\,dx$$



$$y = \frac{1}{x^p} \rightarrow x^p = \frac{1}{y} \rightarrow x = \frac{1}{y^{1/p}}$$

$$= 1 + \int_1^{\infty} \frac{1}{y^{1/p}}\,dy = \begin{cases} \text{CONVERGES IF } \frac{1}{p} > 1 \iff p < 1 \\ \text{DIVERGES IF } \frac{1}{p} \le 1 \iff p \ge 1 \end{cases}$$

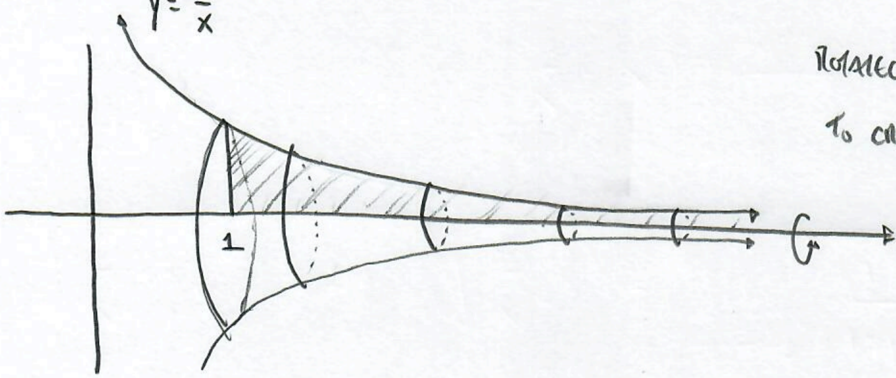
$$\boxed{\int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \text{CONVERGES IF } p < 1 \\ \text{DIVERGES IF } p \ge 1 \end{cases}}$$



$y = \frac{1}{x^p}$, $p > 0$

$y = \frac{1}{x}$ $\quad y = \frac{1}{x^{1/2}}$

$y = \frac{1}{x^2}$



ROTATE CURVE $y=\frac{1}{x}$ AROUND X-AXIS TO CREATE A 3D SHAPE WITH VOLUME V, $1 \le x < \infty$

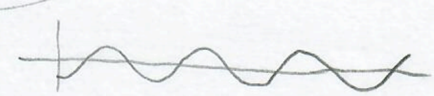
"INFINITE BUGLE"

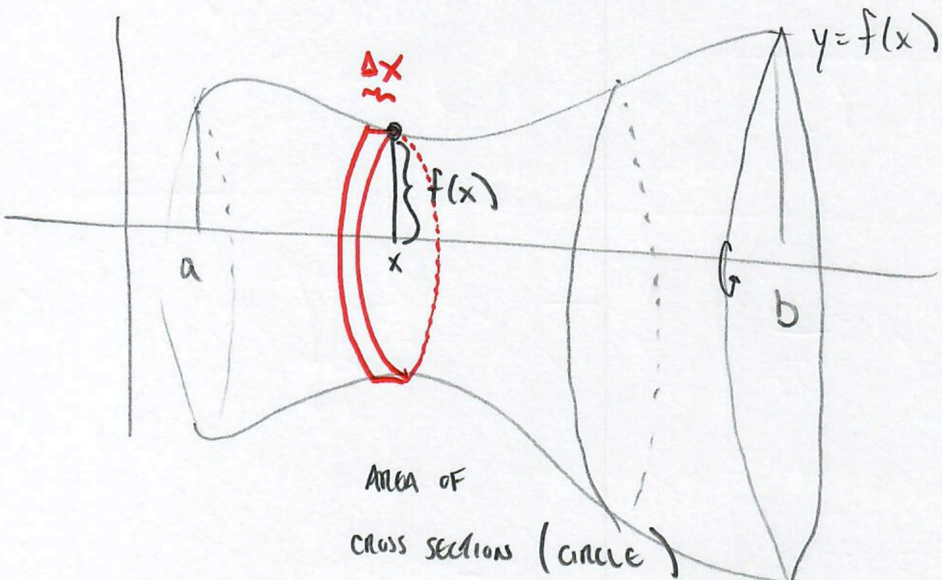
AREA ▨ $= \int_1^\infty \frac{1}{x}\,dx = \infty$ (p-TEST WITH $p=1$)

(i.e. IMP. INT. DIVERGES)

$$\left(\int_1^\infty \sin x \, dx = \underbrace{\lim_{t\to\infty} -\cos t}_{\text{DNE}} + \cos(1) \right)$$

DIVERGES





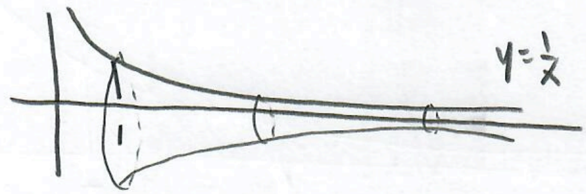
AREA OF CROSS SECTION (CIRCLE)

$\pi f(x)^2$

DISK AS VOLUME $\pi f(x)^2 \Delta x$

$$V = \int_a^b \pi f(x)^2 \, dx$$

VOLUME

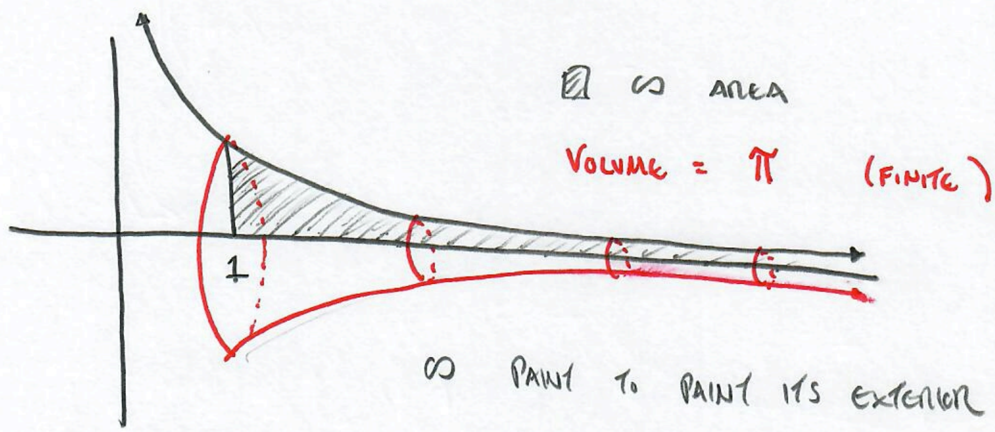


$$V = \int_1^\infty \pi \left(\frac{1}{x}\right)^2 dx = \pi \int_1^\infty \frac{1}{x^2}\, dx$$

CONVERGES

(p - TEST , $p = 2 > 1$)

$$= \frac{\pi}{2-1} = \boxed{\pi}$$



∞ PAINT TO PAINT ITS EXTERIOR

BUT IT CAN BE FILLED PAINT

WITH ONLY π UNITS OF PAINT.

THM. (DIRECT COMPARISON THEOREM)

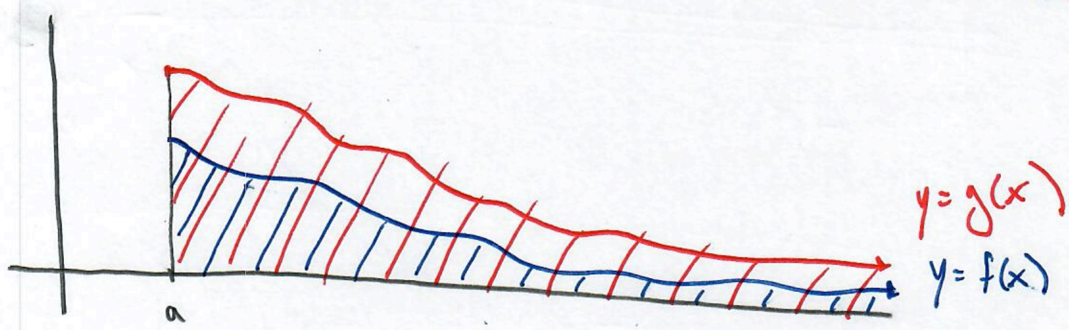
LET f, g CONTINUOUS ON $[a, \infty)$ WITH $0 \le f(x) \le g(x)$ FOR ALL $x \ge a$.

1. IF $\displaystyle\int_a^\infty g(x)\,dx$ CONVERGES THEN $\displaystyle\int_a^\infty f(x)\,dx$

FINITE ALSO FINITE

2. IF $\displaystyle\int_a^\infty f(x)\,dx$ DIVERGES THEN $\displaystyle\int_a^\infty g(x)\,dx$

INFINITE ALSO INFINITE



ex. $\displaystyle\int_\pi^\infty \frac{1+\sin x}{x^2}\,dx$ CONVERGE OR DIVERGE?

NOTE: $-1 \le \sin x \le 1$

$0 \le 1 + \sin x \le 2$

$0 \le \boxed{\dfrac{1+\sin x}{x^2} \le \dfrac{2}{x^2}}$

$\therefore \quad 0 \le \displaystyle\int_\pi^\infty \frac{1+\sin x}{x^2}\,dx \le \int_\pi^\infty \frac{2}{x^2}\,dx \le 2\int_1^\infty \frac{1}{x^2}\,dx$ CONVERGES. (p-TEST)

CONVERGES BY D.C.T.

DOESN'T MATTER (AS LONG AS IT'S POSITIVE)

$$\int_0^\infty \frac{1}{e^x + e^{-x}}\,dx$$

CONVERGE OR DIVERGE ?

DEF.

P-TEST ✗

D.C.T. ✓

COMPARE TO A SIMPLER FUNC.

EITHER LARGER FUNC. THAT CONV.

OR SMALLER FUN. THAT DIV.

OBSERVE: $e^{-x} \geq 0$

$$e^x + e^{-x} \geq e^x$$

$$0 \leq \frac{1}{e^x + e^{-x}} \leq \frac{1}{e^x}$$

$$\left(\begin{array}{c} 2 < 3 \\ \downarrow \\ \frac{1}{2} > \frac{1}{3} \end{array} \right)$$

COMPARISON

SO $$0 \leq \int_0^\infty \frac{1}{e^x + e^{-x}}\,dx \leq \int_0^\infty \frac{1}{e^x}\,dx = \lim_{t \to \infty} \int_0^t e^{-x}\,dx$$

$$= \left(\lim_{t \to \infty} -e^{-t} \right) + \underbrace{e^0}_{1} = 1 \quad \text{CONVERGES}$$

0

$\therefore$ BY D.C.T. $\int_1^\infty \frac{1}{e^x + e^{-x}}\,dx$ ALSO CONVERGES.

THM. (LIMIT COMPARISON TEST)

IF f, g ARE POSITIVE & CONTINUOUS ON $[a, \infty)$ AND

IF $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$, $0 < L < \infty$

(POSITIVE, FINITE #)

THEN $\int_a^\infty f(x)\,dx$ & $\int_a^\infty g(x)\,dx$

$\left(f(x) \approx L\, g(x) \right)$

$\int f(x)\,dx \approx L \int g(x)\,dx$

EITHER BOTH DIVERGE OR BOTH CONVERGE.

ex. $\int_1^\infty \frac{1}{e^x - 1}\,dx$ TRY TO USE D.C.T.

$\frac{1}{e^x - 1} \geq \frac{1}{e^x}$

(SMALLER DENOM $\Rightarrow$ LARGER #'s
LARGER DENOM $\Rightarrow$ SMALLER #'s)

$\therefore \int_1^\infty \frac{1}{e^x-1}\,dx \geq \int_1^\infty \frac{1}{e^x}\,dx = 1$ CONVERGES.

?

COULD BE FINITE, COULD BE INFINITE. D.C.T. INCONCLUSIVE.

TRY LIMIT. COMP. THM (LCT)

$$\lim_{x\to\infty} \frac{\frac{1}{e^x-1}}{\frac{1}{e^x}} = \lim_{x\to\infty} \frac{e^x}{e^x - 1} \overset{\frac{\infty}{\infty},\ \text{L'HO}}{=} \lim_{x\to\infty} \frac{e^x}{e^x} = \underset{L}{1}$$

SINCE $\int_1^\infty \frac{1}{e^x}\,dx$ CONVERGES, BY L.C.T.,

$\int_1^\infty \frac{1}{e^x - 1}\,dx$ ALSO CONVERGES.

EX. $\int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$

CONVERGE OR DIVERGE?

WHY?

- WHAT TEST(S) DID YOU USE?

- DIRECTLY WITH DEF $\lim_{t\to\infty} \int_1^t \ldots$
- DIRECT COMP. THM

$$f(x) \le g(x)$$

- LIMIT COM THM

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty$$

$$\frac{\sqrt{x+1}}{x^2} \ge \frac{\sqrt{x}}{x^2} = \frac{1}{x^{3/2}}$$

SMALLER FUNC WHOSE IMP INT. $\int_1^\infty \frac{1}{x^{3/2}}\,dx$ CONVERGES BY p-TEST

$$\left(p = \frac{3}{2} > 1\right)$$

$$\frac{\sqrt{x+1}}{x^2} \ge \frac{\sqrt{x+1}}{(x+1)^2} = \frac{1}{(x+1)^{3/2}}$$

L.C.T. $f(x) = \dfrac{\sqrt{x+1}}{x^2}$, $g(x) = \dfrac{\sqrt{x}}{x^2}$ $\left(= \dfrac{1}{x^{3/2}} \right.$ WHOSE IMP. INT. CONVERGES. $\left. \right)$

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{\sqrt{x+1}}{\cancel{x^2}} \cdot \frac{\cancel{x^2}}{\sqrt{x}}$$

$$= \lim_{x \to \infty} \sqrt{\frac{x+1}{x}} = \lim_{x \to \infty} \sqrt{1 + \frac{1}{x}} = 1$$

$\therefore$ BY L.C.T. SINCE $\displaystyle\int_1^\infty \frac{\sqrt{x}}{x^2}\,dx = \int_1^\infty \frac{1}{x^{3/2}}\,dx$ CONVERGES

BY p-TEST $\left(p = \frac{3}{2} > 1 \right)$

& $\displaystyle\lim_{x \to \infty} \frac{\frac{\sqrt{x+1}}{x^2}}{\frac{\sqrt{x}}{x^2}} = 1$, $\displaystyle\int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$ ALSO CONVERGES.

QUIZ # 2 TOMORROW (8:35 -9:05) AM ON PEARSON.

9:10 LECTURE AGAIN.

OFFICE HOURS: MW 1:30-3:30 PM