§10.1 SEQUENCES

ORDERED, REAL NUMBERS.

Def: A SEQUENCE IS A LIST OF OBJECTS

1ST OBJECT , 2nd OBJECT , ...

$$a_1 \ , \ a_2 \ , \ a_3 \ , \ \ldots$$

SUBSCRIPTS ARE CALLED INDEX

$$\ldots , \ a_{n-1} \ , \ a_n \ , \ a_{n+1} \ , \ \ldots$$

$(n-1)^{TH}$ TERM $\qquad$ n^{TH} TERM

CONTINUES FOR EVER (∞ LIST)

e.g. SEQUENCE : $2, 4, 6, 8, 10, \ldots$

$$a_1 = 2 \qquad a_2 = 4 \qquad a_3 = 6$$

(RULE) $\underbrace{a_n = 2n}_{\text{FUNCTION!}}$ $\qquad f(n) = 2n$

A SEQUENCE IS A FUNCTION WITH DOMAIN IS $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

ANY EXPRESSION OF n

NOTATION: $\boxed{a_n = (-1)^{n+1}\frac{1}{n}}$ (e.g. TO FIND a_7 PLUG IN $n=7$) $f(x) = \ldots\ldots$

DEFINING A SEQUENCE

$$\{a_n\} = \{\underbrace{1}_{a_1}, \underbrace{-\tfrac{1}{2}}_{a_2}, \tfrac{1}{3}, -\tfrac{1}{4}, \tfrac{1}{5}, \ldots\}$$

$$\{a_n\} = \left\{(-1)^{n+1}\frac{1}{n}\right\}_{n=1}^{\infty}$$

e.g. $a_n = \dfrac{n+2}{n} \rightarrow \underset{n=1}{\overset{a_1}{3}}, \underset{n=2}{\overset{a_2}{2}}, \overset{a_3}{\tfrac{5}{3}}, \tfrac{6}{4}, \ldots$

e.g. $a_n = (-1)^n \dfrac{10^n}{n!}$ "n-FACTORIAL" $n! = n(n-1)(n-2)\cdots 2\cdot 1$

$$a_1 = (-1)\frac{10}{1!} = \ominus 10$$

$$a_2 = (-1)^2 \frac{10^2}{2!} = {}^{+}50$$

$$a_3 = (-1)^3 \frac{10^3}{3!} = \frac{\ominus 1000}{6}$$

$3\cdot 2\cdot 1$

Def: THE SEQUENCE a_n CONVERGES TO THE LIMIT L

IF FOR ALL REAL NUMBERS $\varepsilon > 0$, THERE EXISTS A POSITIVE INTEGER N SUCH THAT

$$|a_n - L| < \varepsilon \quad \text{WHENEVER} \quad n \geq N.$$

$a_1, a_2 \ldots, a_N, a_{N+1}, \ldots$

n^{TH} TERM a_n IS CLOSE TO LIMIT L

WHEN n IS BIG ENOUGH (GO OUT FAR ENOUGH INTO SEQUENCE)

WE WRITE $\lim\limits_{n\to\infty} a_n = L$, OR $a_n \to L$ AS $n \to \infty$.

IF NO LIMIT EXISTS, WE SAY THE SEQUENCE a_n DIVERGES.

e.g. $a_n = \frac{1}{n}$: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

$$\lim_{n\to\infty} a_n = 0$$

FARTHER OUT YOU GO, CLOSER TO 0 THE TERMS BECOME.

THM. CALCULATING LIMITS OF SEQUENCES

Let a_n, b_n ARE TWO SEQUENCES OF REAL NUMBERS.

Let $A, B \in \mathbb{R}$ SUCH THAT $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$.

Let $c \in \mathbb{R}$. "BELONGS TO THE SET OF REAL NUMBERS"

1. $\lim_{n\to\infty} (a_n \pm b_n) = A \pm B$

   (LIMIT OF SUM/DIFF = SUM/DIFF OF LIMITS.)

2. $\lim_{n\to\infty} c a_n = cA$

3. $\lim_{n\to\infty} a_n b_n = AB$

   (LIMIT OF PRODUCT = PRODUCT OF LIMITS)

4. $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{A}{B}$

   (LIMIT OF QUOTIENTS = QUOTIENT OF LIMITS) e.g. FIND LIMIT OF SEQUENCE $a_n = \dfrac{1-5n^4}{n^4+8n^3}$

$$\lim_{n\to\infty} \frac{1-5n^4}{n^4+8n^3} \cdot \frac{1/n^4}{1/n^4} = \lim_{n\to\infty} \frac{\frac{1}{n^4}-5}{1+\frac{8}{n}}$$

$$\overset{(4)}{=} \frac{\lim_{n\to\infty}\left(\frac{1}{n^4}-5\right)}{\lim_{n\to\infty}\left(1+\frac{8}{n}\right)} = \frac{\lim_{n\to\infty}\left(\frac{1}{n^4}\right)-\lim_{n\to\infty}(5)}{\lim_{n\to\infty}(1)+\lim_{n\to\infty}\left(\frac{8}{n}\right)}$$

$$= \frac{0-5}{1+0} = \boxed{-5}$$

e.g. $$\lim_{n\to\infty}\left(\frac{n+1}{2n}\right)\left(\frac{1-5n^4}{n^4+8n^3}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{2n}\right)\lim_{n\to\infty}\left(\frac{1-5n^4}{n^4+8n^3}\right)$$

$$= \left(\frac{1}{2}\right)(-5) = \boxed{-\frac{5}{2}}$$

THM (SANDWICH THM)

LET a_n, b_n, c_n BE SEQUENCES OF REAL NUMBERS.

IF $a_n \leq b_n \leq c_n$ HOLD FOR ALL $\underbrace{n \geq N}$

BEYOND A CERTAIN INDEX.

"EVENTUALLY"

AND IF $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$.

THEN:

$$a_n \leq b_n \leq c_n$$

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n \leq \lim_{n\to\infty} c_n$$

$$L \leq \lim_{n\to\infty} b_n \leq L$$

$$\lim_{n\to\infty} b_n = L$$

$$\lim_{n\to\infty} b_n = L.$$

$= 1, -1, 1, -1, \ldots$ ALTERNATING

ex.

$$a_n = \frac{(-1)^{n+1}}{2^n} \qquad \text{FIND } \lim_{n\to\infty} a_n$$

$$\frac{-1}{2^n} \le a_n \le \frac{1}{2^n}$$

$$0 = \lim_{n\to\infty} \frac{-1}{2^n} \le \lim_{n\to\infty} a_n \le \lim_{n\to\infty} \frac{1}{2^n} = 0$$

$\therefore$ BY SANDWICH THM, $\lim_{n\to\infty} a_n = 0$.

ex.

$$a_n = \frac{\cos(n)}{n^2} \qquad -1 \le \cos(n) \le 1$$

$$\frac{1}{n^2} \le a_n \le \frac{1}{n^2}$$

$$0 = \lim_{n\to\infty} \frac{-1}{n^2} \le \lim_{n\to\infty} a_n \le \lim_{n\to\infty} \frac{1}{n^2} = 0$$

ALSO 0 BY SANDWICH THM.

THM **CONTINUOUS FUNC. THM. FOR SEQUENCES**

LET a_n BE A SEQUENCE OF REAL NUMBERS.

IF $\lim_{n\to\infty} a_n = L$ AND IF f IS CONTINUOUS

AT L THEN

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right)$$

$$= f(L).$$

e.g. $\lim_{n\to\infty} \sqrt{\frac{3n^2+1}{2n^2-1}} = \sqrt{\lim_{n\to\infty} \frac{3n^2+1}{2n^2-1}}$

$$= \sqrt{\frac{3}{2}}$$

THM: SUPPOSE $f(x)$ IS A FUNCTION DEFINED FOR ALL $x \geq N$, WHERE N IS SOME POSITIVE INTEGER.

SUPPOSE $a_n = f(n)$ FOR $n \geq N$.

THEN $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = L$ (ASSUMING THIS LIMIT L EXISTS) ex. $a_n = \left(\frac{n+1}{n-1}\right)^n$. FIND $\lim_{n\to\infty} a_n$.

1^∞ ← INDETERMINATE FORM

NOTE: DEFINE $f(x) = \left(\frac{x+1}{x-1}\right)^x$ A FUNCTION OF A REAL VARIABLE DOMAIN $(f) = \mathbb{R}$

IF $\lim_{x\to\infty} f(x) = L$, THEN $\lim_{n\to\infty} a_n = L$.

$f(n) = a_n$ THIS IS THE RELATION BETWEEN FUNC. $f(x)$ & SEQ. a_n

$$\lim_{x\to\infty} \left(\frac{x+1}{x-1}\right)^x \quad (: 1^\infty)$$

$$= \lim_{x\to\infty} e^{x \ln\left[\left(\frac{x+1}{x-1}\right)^x\right]} = e^{\overbrace{\lim_{x\to\infty}\left(\frac{\ln\left(\frac{x+1}{x-1}\right)}{1/x}\right)}^{\frac{0}{0} \text{ IND. FORM.}}}$$

$$\overset{\text{L'HÔ}}{=} e^{\lim_{n\to\infty}\left(\frac{\frac{1}{x+1} - \frac{1}{x-1}}{-1/x^2}\right)} = e^{\lim_{n\to\infty} \frac{((\cancel{x}-1) - (\cancel{x}+1))(-x^2)}{(x+1)(x-1)}}$$

$$= e^{\lim_{n\to\infty}\left(\frac{2x^2}{x^2-1}\right)} = \boxed{e^2}$$