ex $a_n = \dfrac{n+4}{n}$

$a_1 = 5/1 = 5$
$a_2 = 6/2 = 3$
$a_3 = 7/3$
$a_4 = 6/4 = 1.5$

L. UPPERBOUND 5

G. LOWER BOUND 0

BOUNDED = •) BOUNDED ABOVE ($\exists$ UPPER BOUND)

•) BOUNDED BELOW ($\exists$ LOWER BOUND)

TO SHOW MONOTONICITY:

USUALLY, WE FIND $f(x)$ DIFFERENTIABLE SUCH THAT

$f(n) = a_n$ FOR $n \geq N$.

THEN IF $f'(x) < 0$ $\forall x \geq N$ $\Rightarrow$ SEQ IS DECR. *
(NON-INCRE.)

$f'(x) > 0$ $\forall x \geq N$ $\Rightarrow$ SEQ IS INCR *
(NON-DECR.)

* MONOTONIC.

$$a_n = \frac{5^n 9^n}{n!} = \frac{(5\cdot 9)^n}{n!} = \frac{45^n}{n!}$$

$n(n-1)(n-2)\cdots(2)(1)$

$a_1 = 45$ BIGGER

$a_2 = \frac{45^2}{2!} =$ BIGGER

$a_3 = \frac{45^3}{3!} =$

$$a_1 = \frac{45}{1}$$

$$a_2 = \frac{45}{2} \cdot \frac{45}{1} = \frac{45}{2} \cdot a_1$$

BIGGER

$$a_3 = \frac{45}{3} \cdot \left(\frac{45}{2} \cdot \frac{45}{1} \right) = \frac{45}{3} a_2$$

$$a_{44} = \frac{45^{44}}{44!}$$

$$n! = n\underline{(n-1)!}$$
$$= n(n-1)\underline{(n-2)!}$$

$$a_{45} = \frac{45^{45}}{\boxed{45!}} = \boxed{\frac{45}{45} \cdot \frac{45^{44}}{44!}} = a_{44}$$

SMALLER

$$a_{46} = \frac{45^{46}}{46!} = \frac{45}{46} \cdot \frac{45^{45}}{45!} = \frac{45}{46} \cdot a_{45}$$

<1

$$a_{47} = \frac{45}{47} a_{46}$$

·) LOWER BOUND 0

·) UPPER BOUND $a_{44} = a_{45}$ IS LEAST UPPER BOUND.

·) SEQUENCE IS DECREASING FOR $n \geq 45$

$$a_n = \frac{n+1}{n}$$

0 IS LOWER BOUND

$$\frac{n}{n} + \frac{1}{n} = 1 + \frac{1}{n}$$

1 IS LOWER BOUND

1 IS GREATEST LOWER BOUND.

QUIZ #3 §8.8 IMP. INT. (2 QUEST.)

§10.1 SEQ.

§10.2 SERIES (GEOMETRIC)

§10.3 INT. TEST.

8:35 ~ 35 MIN. LECT. 9:15 AM

§10.4 COMPARISON TESTS. → DIRECT COMPARISON TEST / → LIMIT COMPARISON TEST   IMP. WT. SERIES

QUESTION: DOES THE SERIES $\sum_{n=1}^{\infty} a_n$ CONVERGE OR DIVERGE?

THM (DIRECT COMPARISON TEST)

LET $\sum a_n \left(= \sum_{n=1}^{\infty} a_n \right)$ & $\sum b_n$ BE TWO SERIES SUCH THAT $0 \leq a_n \leq b_n$ FOR ALL n.

THEN
1. IF $\sum b_n$ CONVERGES THEN $\sum a_n$ ALSO CONVERGES.
2. IF $\sum a_n$ DIVERGES THEN $\sum b_n$ ALSO DIVERGES.

PROOF: TERMS ARE ALL NON-NEGATIVE, SO SEQUENCE OF PARTIAL SUMS $\left(s_n = \sum_{i=1}^{n} a_i \right)$ IS NON-DECREASING. THEREFORE THE SERIES (SEQUENCE OF PARTIAL SUMS) CONVERGES IF & ONLY IF IT IS BOUNDED ABOVE.

1. $$\underbrace{a_1 + a_2 + \dots + a_n}_{} \leq b_1 + b_2 + \dots + b_n \leq \sum_{n=1}^{\infty} b_n = M$$

ALL PARTIAL SUMS ARE BOUNDED ABOVE BY M. ✓

2. $$\underbrace{a_1 + a_2 + \dots + a_n}_{\text{UNBOUNDED AS } n \to \infty} \leq \underbrace{b_1 + b_2 + \dots + b_n}_{\text{MUST ALSO BE UNBOUNDED}}$$ ✓

ex. CONVERGE OR DIVERGE?

$$\sum_{n=1}^{\infty} \frac{n-1}{n^4+2}$$

JUST BECAUSE TERMS $\to 0$ DOES NOT IMPLY THAT THE SERIES CONVERGES

(e.g. $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ DIVERGES.

HARMONIC SERIES)

COMPARE:

LARGER NUMERATOR

$$\frac{n-1}{n^4+2} \le \frac{n}{n^4} = \frac{1}{n^3}$$

SMALLER DENOMINATOR

$\boxed{\text{IF TERMS DO NOT GO TO } 0, \text{ THEN THE SERIES DIVERGES.}}$

DIVERGENCE TEST (§10.2)

SINCE $\sum_{n=1}^{\infty} \frac{1}{n^3}$ CONVERGES ($p=3$)

$$\boxed{p\text{-TEST}: \sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{CONVERGES IF } p>1 \\ \text{DIVERGES IF } p \le 1 \end{cases}}$$

(p-TEST §8.8 IMP. INT) (p-SERIES TEST §10.3 INT. TEST)

BY D.C.T. $\sum_{n=1}^{\infty} a_n$ ALSO CONVERGES.

REINDEXING

ex. $\sum_{n=1}^{\infty} \frac{1}{n3^n}$

→ GEOMETRIC SERIES

→ p-SERIES

$\frac{1}{n3^n} \le \left(\frac{1}{3}\right)^n$ $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ CON

$\frac{1}{n3^n} \le \frac{1}{n}$ $\boxed{\sum_{n=1}^{\infty} \frac{1}{n} \text{ DIVERGES}}$

$\boxed{\text{GEO SERIES } \sum_{n=1}^{\infty} ar^{n-1} \text{ CONVERGES WHEN } |r|<1.}$

$$\sum_{n=1}^{\infty} \frac{1}{n3^n} \le \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^{n-1}$$

CONVERGES.

$\left(a = \frac{1}{3} \quad r = \frac{1}{3}\right)$

BY D.C.T. ORIGINAL SERIES ALSO CONV.

ex. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

- DIVERGENCE TEST
- GEOMETRIC
- p - SERIES
- D.C.T.
- INT. TEST.

(DIVERGENCE TEST : IF TERMS $\not\to 0$
THEN SERIES DIVERGES)

$$\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{3\cdot 2} + \frac{1}{4\cdot 3\cdot 2} + \frac{1}{5\cdot 4\cdot 3\cdot 2} + \dots$$

$$\leq 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots$$

(SMALLER DENOM = LARGER FRACTION)

$$= \underbrace{\sum_{n=1}^{\infty} 1\left(\frac{1}{2}\right)^{n-1}}_{(=2)}$$

GEO. SERIES CONVERGES ✓

$a = 1$, $r = \frac{1}{2}$

BY D.C.T. $\sum \frac{1}{n!}$ ALSO CONVERGES.

THM (LIMIT COMPARISON TEST)

SUPPOSE $a_n > 0$ & $b_n > 0$ $\forall n \geq 1$.

1. IF $\lim\limits_{n \to \infty} \frac{a_n}{b_n} = c$ AND $0 < c < \infty$

THEN $\sum a_n$ & $\sum b_n$ EITHER BOTH CONVERGE OR BOTH DIVERGE.

2. IF $\lim\limits_{n \to \infty} \frac{a_n}{b_n} = 0$ AND $\sum b_n$ CONVERGES THEN $\sum a_n$ ALSO CONVERGES.

3. IF $\lim\limits_{n \to \infty} \frac{a_n}{b_n} = \infty$ AND $\sum b_n$ DIVERGE THEN $\sum a_n$ ALSO DIVERGES.

PROOF: 1. SINCE $\lim\limits_{n \to \infty} \frac{a_n}{b_n} = c$ (FOR LARGE n, $\frac{a_n}{b_n} \approx c$, $a_n \approx c b_n$)

SO THERE EXISTS N S.T. FOR $n \geq N$ WE HAVE

$$\underbrace{\left| \frac{a_n}{b_n} - c \right|}_{\text{SHRINKS TO } 0} \leq \frac{c}{2}$$ ← ANY POSITIVE # WILL DO.

AND SO $-\frac{c}{2} \leq \frac{a_n}{b_n} - c \leq \frac{c}{2}$

$+c \qquad + c \qquad + c$

$$\frac{c}{2} \leq \frac{a_n}{b_n} \leq \frac{3c}{2}$$

$$\underbrace{\frac{c}{2} b_n \leq a_n}\ \underbrace{\leq \frac{3c}{2} b_2}$$

IF $\sum b_n$ DIVERGES,

THEN SO DOES $\sum \frac{c}{2} b_n$

$\Rightarrow \sum a_n$ ALSO DIV. BY D.C.T.

IF $\sum b_n$ CONVERGES

THEN SO DOES $\sum \frac{3c}{2} b_2$

$\Rightarrow \sum a_n$ ALSO CONVERGES BY D.C.T.

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ex. $$\sum_{n=1}^{\infty} \left(\frac{2n+3}{5n+4}\right)^n$$

IDEA: LARGE n: $a_n \approx \left(\frac{2}{5}\right)^n$ GEOM. SERIES. $r = \frac{2}{5}$ CONV.

$n = 1{,}000{,}000$:

$$a_n = \left(\frac{2{,}000{,}003}{5{,}000{,}004}\right)^n \approx \left(\frac{2}{5}\right)^n$$

D.C.T. FAILS.

$$\left(\frac{2n+3}{5n+4}\right)^n \leq \left(\frac{2n+3}{5n}\right)^n \geq \left(\frac{2\cancel{n}}{5\cancel{n}}\right)^n$$

L.C.T. $$a_n = \left(\frac{2n+3}{5n+4}\right)^n$$

GIVEN

$$b_n = \left(\frac{2}{5}\right)^n$$

CHOSEN

BE ABLE TO ANSWER:

DOES $\sum b_n$ CONV. OR DIV.

$\sum b_n$ CONV.

$$\sum_{n=1}^{\infty} \frac{2}{5} \cdot \left(\frac{2}{5}\right)^{n-1}$$

$a = \frac{2}{5}$, $r = \frac{2}{5}$

CONV.

$$\lim_{n\to\infty} \frac{a_n}{b_n} =$$

$$= \lim_{n\to\infty} \frac{\left(\frac{2n+3}{5n+4}\right)^n}{\left(\frac{2}{5}\right)^n} = \lim_{n\to\infty} \left(\frac{2n+3}{5n+4} \cdot \frac{5}{2}\right)^n$$

$$= \lim_{n\to\infty} \left(\frac{10n+15}{10n+8}\right)^n \quad : \quad 1^{\infty}$$

$$= e^{\ln\left(\lim_{n\to\infty}\left(\frac{10n+15}{10n+8}\right)^{n}\right)}$$

$$= e^{\lim_{n\to\infty} n \cdot \ln\left(\frac{10n+15}{10n+8}\right)} \quad : \quad \infty \cdot 0$$

$$= e^{\lim_{n\to\infty} \frac{\ln\left(\frac{10n+15}{10n+8}\right)}{1/n}} \quad : \quad \frac{0}{0}$$

L'Hô

$$= e^{\lim_{n\to\infty} \frac{\frac{10}{10n+15} - \frac{10}{10n+8}}{-1/n^2}}$$

$$= e^{\lim_{n\to\infty} -10n^2\left(\frac{1}{10n+15} - \frac{1}{10n+8}\right)}$$

$$= e^{\lim_{n\to\infty} -10n^2\left(\frac{\cancel{10n}+8-(\cancel{10n}+15)}{(10n+15)(10n+8)}\right)}$$

$$= e^{\lim_{n\to\infty} \boxed{\frac{-10n^2(-7)}{100n^2+230n+120}}} = e^{7/10} \quad \left(0 < e^{7/10} < \infty\right)$$

L.C.T. CASE 1: $\sum \left(\frac{2n+3}{5n+4}\right)^n$ SAME BEHAVIOR AS $\sum \left(\frac{2}{5}\right)^n$

$\Rightarrow$ CONVERGES BY L.C.T.

ex. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 3^n}{3^n + 4^n}$

IDEA: FOR LARGE n, LOOK AT LARGEST TERMS IN NUM. & DENOM.

$$\underbrace{\frac{2^n + 3^n}{3^n + 4^n}}_{a_n} \approx \underbrace{\left(\frac{3}{4}\right)^n}_{b_n}$$

$\sum \left(\frac{3}{4}\right)^n$ GEOM. SERIES CONV.

$|r| < 1$, $r = \frac{3}{4}$

L.C.T. $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n + 3^n}{3^n + 4^n} \cdot \frac{4^n}{3^n}$

$$= \lim_{n\to\infty} \frac{8^n + 12^n}{9^n + 12^n} \quad \begin{matrix} \div 12^n \\ \div 12^n \end{matrix}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{8}{12}\right)^n + 1}{\left(\frac{9}{12}\right)^n + 1} = 1.$$

$0 < 1 < \infty$

$\therefore$ BY L.C.T. $\displaystyle\sum \frac{2^n + 3^n}{3^n + 4^n}$ & $\displaystyle\sum \left(\frac{3}{4}\right)^n$

<u>BOTH</u> CONVERGE.