§10.5 ABSOLUTE CONVERGENCE, RATIO TEST, ROOT TEST

Def: A SERIES $\sum a_n$ CONVERGES ABSOLUTELY IF

$\sum |a_n|$ CONVERGES.

e.g. $\frac{2}{5} - \frac{4}{25} + \frac{8}{125} - \frac{16}{625} + \dots$

CONVERGES (GEOMETRIC $\sum_{n=1}^{\infty} \frac{2}{5}\left(-\frac{2}{5}\right)^{n-1} = \frac{2/5}{1-(-2/5)} = \frac{2}{5} \cdot \frac{5}{7} = \boxed{\frac{2}{7}}$

$a = \frac{2}{5} \quad r = -\frac{2}{5}$

$$a_n = \frac{2}{5}\left(-\frac{2}{5}\right)^{n-1} \qquad |a_n| = \frac{2}{5}\left(\frac{2}{5}\right)^{n-1}$$

$r = \frac{2}{5}$

$$\sum |a_n| = \sum \frac{2}{5}\left(\frac{2}{5}\right)^{n-1}$$

$$= \frac{2/5}{1 - 2/5} = \boxed{\frac{2}{3}}$$

BIGGER

EVEN WITH ALL TERMS POSITIVE,

SERIES STILL CONVERGES

(ABSOLUTE CONVERGES / CONVERGES ABSOLUTELY) e.g. $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

a_n

DOES THIS CONVERGE? (DELAYED)

DOES THE SERIES CONVERGE ABSOLUTELY?

$$a_n = \frac{(-1)^{n+1}}{n}, \quad |a_n| = \frac{1}{n}$$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$$ "HARMONIC SERIES" DIVERGES

(- INT. TEST
- p-SERIES TEST)

THM: (ABSOLUTE CONVERGENCE TEST)

IF $\sum_{n=1}^{\infty} |a_n|$ CONVERGES THEN $\sum a_n$ CONVERGES.

ABSOLUTE CONVERGE $\Longrightarrow$ CONVERGENCE.

i.e. SUPPOSE $a_n \geq 0$ FOR ALL n,

AND $a_1 + a_2 + a_3 + a_4 + \cdots$ CONVERGES.

THEN $\pm a_1 \pm a_2 \pm a_3 \pm a_4 \pm a_5 \pm \cdots$

STILL CONVERGES!

PROOF: FOR ALL n, $-|a_n| \le a_n \le |a_n|$

$$\Rightarrow 0 \le a_n + |a_n| \le 2|a_n|$$

$\sum |a_n|$ CONV.

$\sum 2|a_n|$ CONV.

SO BY D.C.T.

$\sum (a_n + |a_n|)$ CONVERGES.

THEN $\sum a_n = \sum \left((a_n + |a_n|) - |a_n| \right)$

$= \sum (a_n + |a_n|) - \sum |a_n|$ AS LONG AS BOTH CONVERGE.

BOTH CONVERGE!

$\therefore \sum a_n$ CONVERGES. ✓

EX. $\sum_{n=1}^{\infty} \frac{\sin(n)}{n\sqrt{n}}$ ← CANNOT APPLY D.C.T. TO $\sum a_n$ BECAUSE $a_n \not\geq 0$

CONSIDER $\sum_{n=1}^{\infty} \underbrace{\left| \frac{\sin(n)}{n\sqrt{n}} \right|}_{|a_n|}$ $\quad 0 \le |a_n| = \frac{|\sin(n)|}{n\sqrt{n}} \le \frac{1}{n\sqrt{n}}$

SINCE $\sum \frac{1}{n\sqrt{n}} = \sum \frac{1}{n^{3/2}}$ CONVERGES BY P-TEST $p = \frac{3}{2} > 1$

So, $\sum \frac{\sin(n)}{n\sqrt{n}}$ CONVERGES ABSOLUTELY.

$\Rightarrow$ SERIES CONVERGES (ABS. CONV. THM).

2 TESTS FOR ABSOLUTE CONVERGENCE (ABS CONV $\Rightarrow$ CONV.)

RATIO TEST: LET $\sum a_n$ BE ANY SERIES AND SUPPOSE

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho .$$

"RHO"

R FOR RATIO

THEN $\sum a_n$ $\begin{cases} \text{CONVERGES ABSOLUTELY IF } \rho < 1 \\ \text{DIVERGES IF } \rho > 1 \\ \text{INCONCLUSIVE IF } \rho = 1 \end{cases}$

PROOF: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$ MEANS THERE IS SOME N

SUCH THAT FOR $n \geq N$

WE HAVE $\left| \frac{a_{n+1}}{a_n} \right| \approx \rho \Rightarrow \underbrace{|a_{n+1}| \approx \rho |a_n|}_{\approx \text{ GEOMETRIC}}$

$$\sum_{n=1}^{\infty} |a_n| = \underbrace{\sum_{n=1}^{N-1} |a_n|}_{\text{FINITE}} + \overbrace{\sum_{n=N}^{\infty} |a_n|}^{|a_N| + \rho|a_N| + \rho^2|a_N| + \dots}$$

GEOMETRIC

$$\approx \underbrace{\sum_{n=1}^{\infty} |a_N| \rho^{n-1}}$$

CONVERGES WHEN $\rho < 1$ ✓

IF $\rho > 1$ THEN $|a_{n+1}| \approx \rho |a_n|$ TERMS GET BIGGER,

$a_n \not\to 0$, $\sum |a_n|$ DIVERGES BY DIV. TEST.

IF $\rho = 1$: INCONCLUSIVE :

$$\sum \frac{1}{n} : \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1/(n+1)}{1/n} = 1$$

DIVERGES (p - TEST)

$$\sum \frac{1}{n^2} : \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2} = 1$$

CONVERGES (p - TEST)

CONVERGES ABSOLUTELY !

$\Downarrow$

ex. $\displaystyle \underbrace{\sum_{n=1}^{\infty} (-1)^n \frac{n+2}{3^n}}$ CONV OR DIV ?

$$a_n = (-1)^n \frac{n+2}{3^n} \longrightarrow |a_n| = \frac{n+2}{3^n}$$

RATIO TEST : $\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \underbrace{\frac{(n+1)+2}{3^{\cancel{n+1}}}}_{|a_{n+1}|} \cdot \underbrace{\frac{\cancel{3^n}}{n+2}}_{\frac{1}{|a_n|}} \right|$

$$\rho = \lim_{n\to\infty} \frac{1}{3} \cdot \frac{n+3}{n+2} = \frac{1}{3} \underbrace{\lim_{n\to\infty} \frac{n+3}{n+2}}_{1} = \frac{1}{3}$$

$\rho = \frac{1}{3} < 1 \Rightarrow$ SERIES CONVERGES ABSOLUTELY BY RATIO TEST.

ex. CONV. OR DIV? $\displaystyle\sum_{n=1}^{\infty} \cancel{(-1)^n} \frac{n^2 (n+2)!}{n!\, 3^{2n}}$

RATIO TEST: $\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$= \lim_{n\to\infty} \frac{(n+1)^2 \overbrace{(n+1+2)!}^{(n+3)\cancel{(n+2)!}}}{\underbrace{(n+1)!}_{(n+1)\cancel{n}\cancel{(n-1)}\cancel{(n-2)}\cdots\cancel{(2)}\cancel{(1)}}\,[3^2]^{(n+1)}} \cdot \frac{\cancel{n!}\,[\cancel{3^{2n}}]}{n^2\,\cancel{(n+2)!}}$$

$\cancel{n}\cancel{(n-1)}\cancel{(n-2)}\cdots\cancel{(2)}\cdot\cancel{(1)}$

$$= \lim_{n\to\infty} \frac{(n+1)^{\cancel{2}}(n+3)}{\cancel{(n+1)}\,3^2} \cdot \frac{1}{n^2}$$

$$= \frac{1}{9} \underbrace{\lim_{n\to\infty} \frac{(n+1)(n+3)}{n^2}}_{1} = \frac{1}{9} < 1$$

SO SERIES CONVERGES ABSOLUTELY.

THE ROOT TEST:

LET $\sum a_n$ BE ANY SERIES AND SUPPOSE

$$\lim_{n \to \infty} \sqrt[n]{|a_n|} = \rho .$$

THEN $\sum a_n$ $\begin{cases} \text{CONVERGES ABSOLUTELY IF } \rho < 1 \\ \text{DIVERGES IF } \rho > 1 \\ \text{INCONCLUSIVE IF } \rho = 1 . \end{cases}$

HW §10.5