§10.6 ALTERNATING SERIES & COND. CONV.

ex. $\sum_{n=1}^{\infty} \frac{(-2)^{n+1}}{n+5^n}$ DIVERGE? CONVERGE? $\xrightarrow[\text{YES}]{\text{IF}}$ ABSOLUTELY / CONDITIONALLY

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^{n+1}}{n+5^n} \qquad 2^{n+1} = 2 \cdot 2^n$$

ALT SERIES: ALT. SERIES CONVERGES IF $\lim_{n\to\infty} |a_n| = 0$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{2^{n+1}}{n+5^n}$$

FOR ALL $n = 1, 2, \ldots$

SANDWICH THM.:

$$0 \le \frac{2^{n+1}}{n+5^n} \le \frac{2^{n+1}}{5^n} = 2 \cdot \left(\frac{2}{5}\right)^n$$

$$0 \le \underbrace{\lim_{n\to\infty} \frac{2^{n+1}}{n+5^n}}_{\downarrow \; 0} \le \underbrace{\lim_{n\to\infty} 2 \cdot \left(\frac{2}{5}\right)^n}_{0}$$

SERIES CONVERGES!

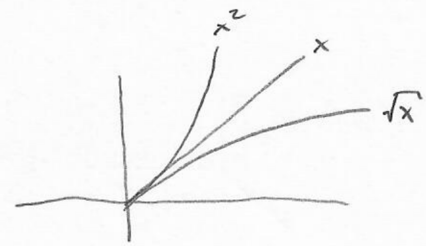
ABSOLUTE CONVERGENCE: $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{2^{n+1}}{n+5^n}$

- D.C.T.
- L.C.T.
- INT. TEST
- RATIO TEST
- n^{th} ROOT TEST

$$\frac{2^{n+1}}{n+5^2} \le 2\left(\frac{2}{5}\right)^n \quad \text{SINCE} \quad \sum_{n=1}^{\infty} 2\left(\frac{2}{5}\right)^n$$

CONVERGES (GEOM. SERIES WITH $r = \frac{2}{5} < 1$)

ORIGINAL SERIES CONVERGES ABSOLUTELY BY DIRECT COMP. TEST.



NOTE: $\lim_{n\to\infty} n^p = \infty$ IF $p > 0$

$\lim_{n\to\infty} \frac{1}{n^p} = 0$ IF $p > 0$

$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^p}$ (ALT SERIES) CONVERGES IF $p > 0$.

MORE PRECISELY:

p-TEST FOR ALT. SERIES:

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^p} \begin{cases} \text{DIVERGES IF } p \le 0 \\ \text{CONVERGES CONDITIONALLY IF } 0 < p \le 1 \\ \text{CONVERGES ABSOLUTELY IF } p > 1. \end{cases}$$

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p-TEST

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{CONVERGES IF } p > 1 \\ \text{DIVERGES IF } p \le 1 \end{cases}$$

CHALLENGE:

DOES THIS LIMIT EXISTS: $\lim_{n\to\infty} \sqrt[n]{n!}$ $= \infty$

$\sum_{n=1}^{\infty} \frac{1}{n!}$ DOES THIS CONVERGE OR DIVERGE?

RATIO TEST: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1}{(n+1)!} \cdot n!$

$= \lim_{n\to\infty} \frac{\cancel{n!}}{(n+1)\cancel{n!}} = 0$ $\therefore$ SERIES CONVERGES BY RATIO TEST.

COULD YOU USE THE ROOT TEST?

$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n!}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}} = ?$

MOVING ON ... NO MORE TESTS FOR CONVERGENCE.

§ 10.7 Power Series (OMIT MULTIPLICATION OF POWER SERIES)

POLYNOMIALS : N^{TH} DEGREE POLYNOMIAL

$$P(x) = \sum_{n=0}^{N} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots c_N x^N$$

THE c_n ARE COEFFICIENTS (CONSTANTS)

c_0 IS THE CONSTANT TERM.

$$\left(P(x) = \sum_{n=0}^{N} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots + c_N(x-a)^N \right.$$

SAME POLYNOMIAL SHIFTED HORIZONTALLY BY a)

∞ DEGREE POLYNOMIAL ??

Def: A POWER SERIES ABOUT $x=0$ IS A SERIES OF THE FORM

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots$$

A POWER SERIES ABOUT $x=a$ IS A SERIES OF THE FORM

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2$$

(HORIZONTAL SHIFT BY a)

WE CALL a THE CENTER OF THE POWER SERIES, & THE CONSTANTS c_n ARE CALLED COEFFICIENTS.

ex. POWER SERIES ABOUT $x=0$ WITH ALL COEFFICIENTS EQUAL TO 1

↳ POLYNOMIAL ARE DETERMINED BY THEIR COEFFICIENTS.

EVERY QUADRATIC POLY: $\underline{a}x^2+\underline{b}x+\underline{c}$

↳ POWER SERIES IS DETERMINED BY COEFFICIENTS.

$c_n = 1$ FOR ALL n.

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \dots$$

A FUNCTION IS A RULE THAT ASSIGNS OUTPUT TO (ACCEPTABLE) INPUT.

$f(x) = \sum_{n=0}^{\infty} x^n$ GEOMETRIC SERIES $\sum_{n=1}^{\infty} ar^{n-1}$ $a = 1$, $r = x$

↳ CONVERGES TO $\dfrac{1}{1-r} = \dfrac{1}{1-x}$ IF $|r| = |x| < 1$

DIVERGE IF $|x| \geq 1$

IN OTHER WORDS $\underbrace{\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n}$ IF $|x| < 1$

2 WAYS TO EXPRESS THE SAME FUNCTION OF x ON INTERVAL $(-1, 1)$ *

* BUT IN GENERAL, THEY ARE DIFFERENT FUNCTIONS, BECAUSE THEY HAVE DIFFERENT DOMAINS.

$$\text{DOM}\left(\frac{1}{1-x}\right) = (-\infty, 1) \cup (1, \infty)$$

$$\text{DOM}\left(\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots\right) = (-1, 1)$$

AGREE ON INTERSECTION OF DOMAINS

ex. FOR WHAT VALUES OF X DOES THE POWER SERIES ABOUT $x = 0$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$ CONVERGE? $\boxed{\text{CERTAINLY } 0}$ CENTER

(LET $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. FIND DOMAIN OF f.)

TEST FOR CONVERGENCE: RATIO TEST:

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$$

$$\rho = \lim_{n\to\infty} \left|\frac{x}{n+1}\right|$$

IF $x = 1$

IF $x = 100$

$$= 0$$ NO MATTER WHAT X IS.

RATIO TEST SAYS SERIES CONVERGES WHEN $\rho < 1$.

THEREFORE THE SERIES CONVERGES FOR ALL X,

$$-\infty < x < \infty, \quad (-\infty, \infty).$$

$$\underbrace{\sum_{n=0}^{\infty} c_n (x-a)^n}_{} \; : \quad c_0 = 0 \, ; \; c_n = \frac{1}{\sqrt{n}} \, , n \geq 1$$

ex. FOR WHAT VALUES DOES THE POWER SERIES ABOUT $x = 1$ CONVERGE?

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} (x-1)^n$$

ROOT TEST:
$$\rho = \lim_{n\to\infty} |a_n|^{1/n} \quad \left(\begin{array}{ll} \text{SERIES CONV. IF} & \rho < 1 \\ \text{SERIES DIV. IF} & \rho > 1 \\ \text{INCONCLUSIVE IF} & \rho = 1 \end{array} \right)$$

$$\rho = \lim_{n\to\infty} \left| \frac{(x-1)^n}{\sqrt{n}} \right|^{1/n} = \lim_{n\to\infty} \frac{\left(|x-1|^{\cancel{n}}\right)^{\cancel{1/n}}}{\sqrt{n^{1/n}}}$$

$$= |x-1| \lim_{n\to\infty} \frac{1}{\sqrt{n^{1/n}}} = |x-1| \cdot \frac{1}{\sqrt{\underbrace{\lim_{n\to\infty} n^{1/n}}_{1}}}$$

$$\rho = |x-1|$$

SERIES CONVERGES IF $|x-1| < 1$

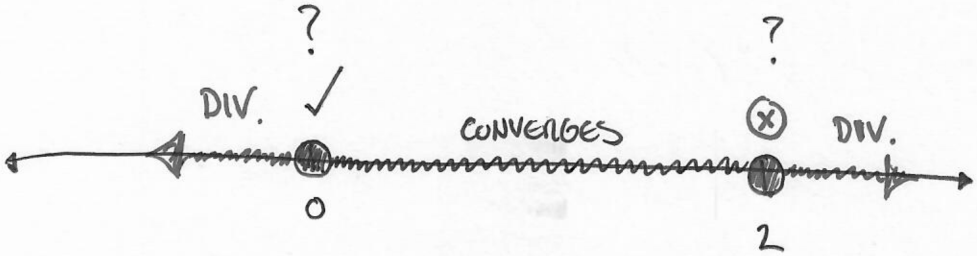
$$-1 < x-1 < 1$$
$$0 < x < 2$$

SERIES DIVERGES IF $|x-1| > 1$

$$x < 0 \quad \text{OR} \quad x > 2$$

INCONCLUSIVE IF $|x-1| = 1$

$$x = 0 \quad \text{OR} \quad x = 2$$



TEST FOR CONVERGENCE AT $x=0$ & $x=2$ SEPARATELY.

$x=0$: $$\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$ ALTERNATING SERIES. TERMS $\to 0$

$\therefore$ CONVERGES AT $x=0$

$x=2$: $$\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$ p-SERIES

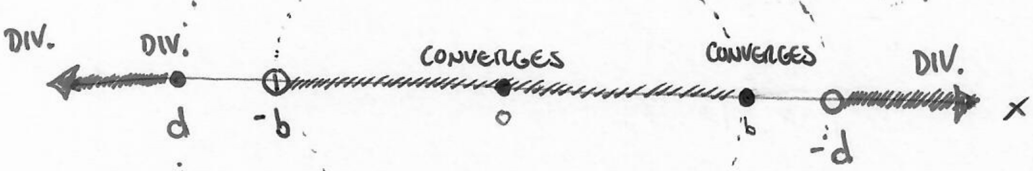
$p=\frac{1}{2} < 1 \Rightarrow$ DIVERGES.

SERIES CONVERGES ON INTERVAL $\underline{[0, 2)}$

THM (CONVERGENCE THEOREM FOR POWER SERIES)

1. IF THE POWER SERIES $\sum_{n=0}^{\infty} c_n x^n$ CONVERGES AT $x=b \neq 0$

   THEN IT CONVERGES FOR ALL VALUES x S.T. $|x| < |b|$



2. IF THE SERIES DIVERGES AT $x=d$,

   THEN THE SERIES DIVERGES FOR ALL x S.T. $|x| > d$.

PROOF: $\boxed{\text{PART 1}}$

SUPPOSE $\sum_{n=0}^{\infty} c_n x^n$ CONVERGES AT $x = b \neq 0$

$$\Rightarrow \sum_{n=0}^{\infty} c_n b^n \text{ CONVERGES.} \Rightarrow \lim_{n\to\infty} c_n b^n = 0 \quad \text{(DIV. TEST)}$$

$\therefore$ THERE EXIST AN INTEGER N S.T. WHEN $n \geq N$ WE HAVE

$$|c_n b^n| < 1 \Rightarrow |c_n| < \frac{1}{|b|^n}.$$

NOW LET $|x| < |b|$, SO $\left|\frac{x}{b}\right| < 1$ *.

MULTIPLY BOTH SIDES BY $|x|^n$

THEN $$|c_n x^n| < \underbrace{\left|\frac{x}{b}\right|^n}$$

$\sum_{n=0}^{\infty} \left|\frac{x}{b}\right|^n$ GEOMETRIC SERIES, CONVERGES $\left(r = \left|\frac{x}{b}\right| < 1\right)$ *

$\therefore$ BY D.C.T. $\sum_{n=0}^{\infty} |c_n x^n|$ CONVERGES ALSO

$$\Rightarrow \sum_{n=0}^{\infty} c_n x^n \text{ CONVERGES ABSOLUTELY } (\text{FOR } |x| < |b|)$$

✓

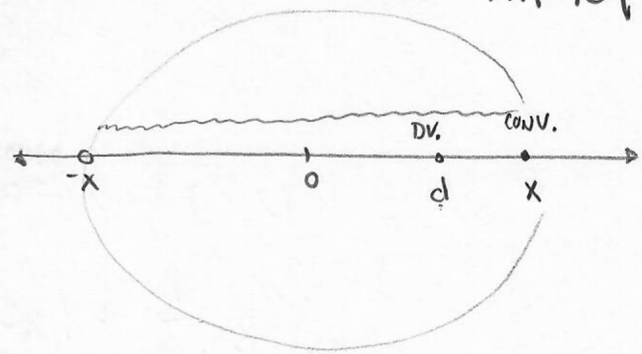
PART 2

SUPPOSE $\sum_{n=0}^{\infty} c_n x^n$ DIVERGES AT $x = d$

NOW ASSUME THE SERIES CONVERGES AT x WITH $|x| > |d|$.

THEN PART 1

$\Downarrow$



SERIES CONVERGES AT d.

$\Rightarrow \Leftarrow$ CONTRADICTION!

THUS, OUR ASSUMPTION THAT SERIES CONVERGES AT SOME POINT x WITH $|x| > |d|$ MUST BE FALSE.

$\therefore$ SERIES MUST DIVERGE AT ALL POINTS x WITH $|x| > |d|$.

✓

MIDTERM EVERYTHING THROUGH §10.6