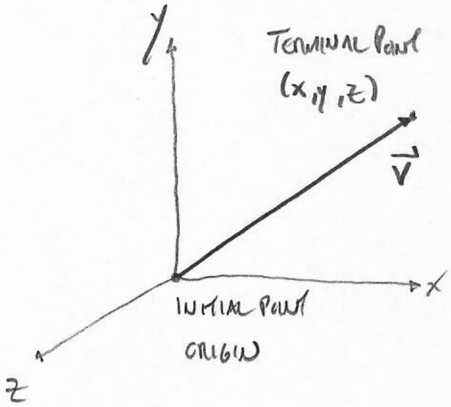
§12.2 Vectors



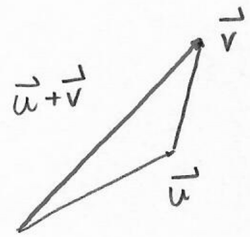
- Length / Magnitude

- Direction

Components of $\vec{v}$

$$\vec{v} = \langle x, y, z \rangle$$

Vector Addition w/ components

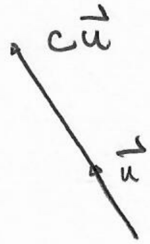


$$\vec{u} = \langle x_1, y_1, z_1 \rangle$$

$$\vec{v} = \langle x_2, y_2, z_2 \rangle$$

$$\vec{u}+\vec{v} = \langle \underbrace{x_1 + x_2, y_1 + y_2, z_1 + z_2}_{\text{ADD EACH COMPONENT}} \rangle$$

Scalar Multiplication



$$\vec{u} = \langle x, y, z \rangle$$

$$c\vec{u} = \langle cx, cy, cz \rangle$$

Not a vector ($\mathbb{R}$)

Note: $\langle 4, 8, 12 \rangle = 4\langle 1, 2, 3 \rangle$

Factor out a scalar

PROPERTIES OF VECTORS (ALGEBRAIC)

IF $\vec{a}, \vec{b}, \vec{c}$ ARE n - DIMENSIONAL VECTORS $(n=2,3)$, c, d SCALARS

(i) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (COMMUTATIVE)

(ii) $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ (ASSOCIATIVE)

(iii) $\vec{a} + \vec{0} = \vec{a}$ (ADDITIVE IDENTITY)

(iv) $\vec{a} - \vec{a} = \vec{0}$ (ADDITIVE INVERSE)

↑ VECTOR

(v) $c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$ (DISTRIBUTIVE, SCALAR)

(vi) $(c + d)\vec{a} = c\vec{a} + d\vec{a}$ (DISTRIBUTIVE, VECTOR)

(vii) $(cd)\vec{a} = a(d\vec{a})$ (ASSOCIATIVE)

(viii) $1\vec{a} = \vec{a}$ (MULTIPLICATIVE IDENTITY)

STANDARD BASIS VECTORS

$\hat{i} = \langle 1, 0, 0 \rangle$ $\hat{j} = \langle 0, 1, 0 \rangle$ $\hat{k} = \langle 0, 0, 1 \rangle$

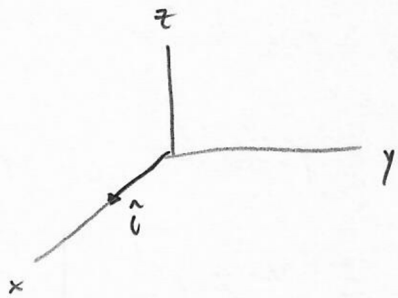
POINTS IN DIRECTION OF POS. X-AXIS, (Y-AXIS) (Z-AXIS)

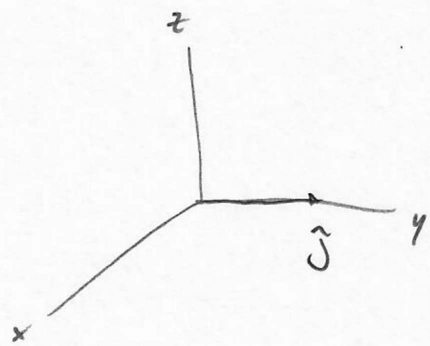
$\underbrace{\text{LENGTH} = 1}$

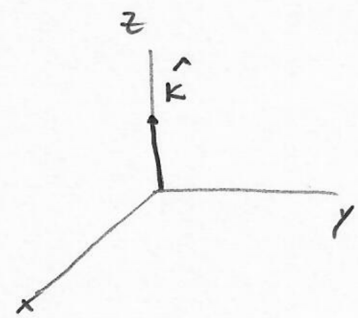
UNIT VECTOR

IN 2 DIM: $\hat{i} = \langle 1, 0 \rangle$

$\hat{j} = \langle 0, 1 \rangle$





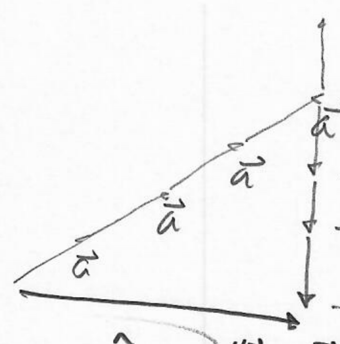


$\therefore \quad \vec{v} = \langle x, y, z \rangle = \langle x, 0, 0 \rangle + \langle 0, y, 0 \rangle + \langle 0, 0, z \rangle$

$= x\langle 1, 0, 0 \rangle + y\langle 0, 1, 0 \rangle + z\langle 0, 0, 1 \rangle$

$= x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{v} = \langle x, y \rangle = x\hat{i} + y\hat{j}$



ex. IF $\vec{a} = 7\hat{i} + 2\hat{j} - 3\hat{k}$, $\vec{b} = 6\hat{j} - \hat{i} + 8\hat{k}$

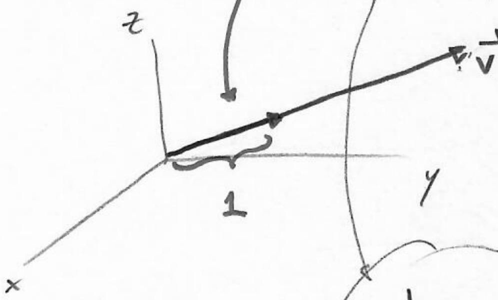
FIND $\boxed{4\vec{a} - 3\vec{b}} = 4\langle 7, 2, -3 \rangle - 3\langle -1, 6, 8 \rangle$ $\boxed{31\hat{i} - 10\hat{j} - 36\hat{k}}$

$= \langle 28, 8, -12 \rangle + \langle 3, -18, -24 \rangle = \boxed{\langle 31, -10, -36 \rangle}$

Def: UNIT VECTOR IS A VECTOR WITH MAGNITUDE/LENGTH 1.

$\therefore$ GIVEN ANY VECTOR $\vec{v} = \langle x, y, z \rangle$

THE UNIT VECTOR WITH THE SAME DIRECTION AS $\vec{v}$ ("NUETRAL" LENGTH)



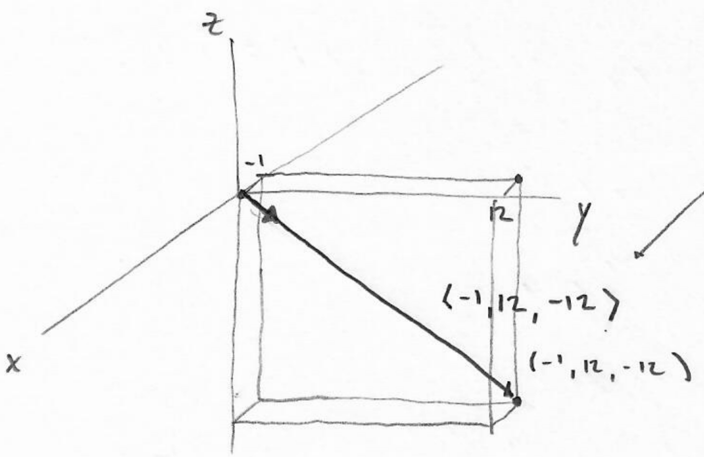
$\vec{v}$ HAS LENGTH $|\vec{v}|$

WE WANT TO SCALE $\vec{v}$ SO IT HAS LENGTH 1.

$\rightarrow \dfrac{1}{|\vec{v}|}\vec{v}$ SCALED BY $\dfrac{1}{\text{LENGTH}}$

ex. FIND THE UNIT VECTOR IN THE SAME DIRECTION AS

$\langle -1, 12, -12 \rangle$



$\langle -1, 12, -12 \rangle$

$\text{LENGTH} = \sqrt{(-1)^2 + (12)^2 + (-12)^2}$

$= \sqrt{289} = 17$

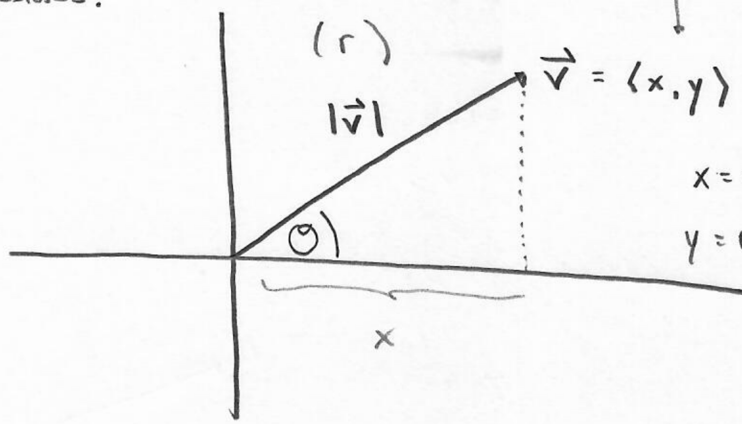
$$\frac{1}{|\vec{v}|}\vec{v} \longrightarrow \boxed{\frac{1}{17}\langle -1, 12, -12 \rangle} = \boxed{\left\langle -\frac{1}{17}, \frac{12}{17}, -\frac{12}{17} \right\rangle}$$

UNIT VECTORS

ANALYTIC

GEOMETRIC

IN 2 DIMENSIONS:



$$\vec{v} = \langle x, y \rangle = \underbrace{\langle |\vec{v}|\cos\theta, |\vec{v}|\sin\theta \rangle}_{}$$

$x = r\cos\theta$

$y = r\sin\theta$

ex. GIVE COMPONENTS OF THE VECTOR WITH LENGTH 5 THAT MAKES ANGLE $\frac{2\pi}{3}$ WITH THE POS. X-AXIS.

$$\langle x, y \rangle = \langle |\vec{v}|\cos\theta, |\vec{v}|\sin\theta \rangle$$

$$= \langle 5\cos\frac{2\pi}{3}, 5\sin\frac{2\pi}{3} \rangle$$

$$= \boxed{\langle -\frac{5}{2}, \frac{5\sqrt{3}}{2} \rangle}$$

- ADD VECTORS ✓
- MULTIPLY BY SCALARS ✓

$\vec{u} - \vec{v} = \vec{u} + (-1\vec{v})$

SUBTRACT ✓

$\vec{a} \div c = \frac{1}{c}\vec{a}$ ✓

HOW TO MULTIPLY 2 VECTORS? $\vec{u}\vec{v}$ ✗

TWO WAYS TO MULTIPLY VECTORS! DOT PRODUCT & CROSS PRODUCT.

§12.3 THE DOT PRODUCT

Def: (ANALYTIC) GIVEN 2 VECTORS $\vec{a} = \langle x_1, y_1, z_1 \rangle$

$\vec{b} = \langle x_2, y_2, z_2 \rangle$

THE THE DOT PRODUCT (SCALAR PRODUCT) OF $\vec{a}$ & $\vec{b}$

$$\vec{a} \cdot \vec{b} = \langle x_1, y_1, z_1 \rangle \cdot \langle x_2, y_2, z_2 \rangle$$

$$= \underbrace{x_1 x_2 + y_1 y_2 + z_1 z_2}$$

NOT A VECTOR, SCALAR.

IN 2D: $\langle x_1, y_1 \rangle \cdot \langle x_2, y_2 \rangle = x_1 x_2 + y_1 y_2$

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ex. $\langle 3, -4, 1 \rangle \cdot \langle -2, 7, 4 \rangle = (3)(-2) + (-4)(7) + (1)(4)$

$= -6 - 28 + 4$

$= \boxed{-30}$

PROPERTIES OF DOT PRODUCT

LET $\vec{a}, \vec{b}, \vec{c}$ BE VECTORS, LET c BE SCALAR

(i) $\vec{a} \cdot \vec{a} = |\vec{a}|^2$

$$\langle x, y, z \rangle \cdot \langle x, y, z \rangle = x^2 + y^2 + z^2$$

$$|\vec{a}|^2 = \left(\sqrt{x^2 + y^2 + z^2}\right)^2$$

✓

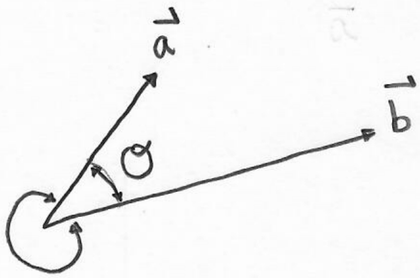
(ii) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

(iii) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

(iv) $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b})$

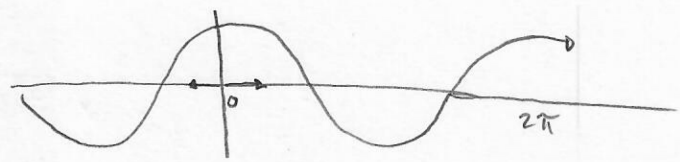
(v) $\vec{0} \cdot \vec{a} = 0$ (COMPARE $0\vec{a} = \vec{0}$)

Geometric Definition of Dot Product



$$\boxed{\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta}$$

WHERE θ IS THE ANGLE BETWEEN $\vec{a}$, $\vec{b}$

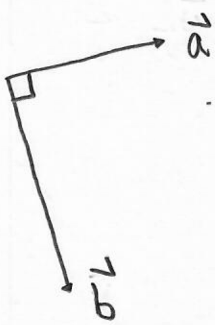


$\cos$ EVEN $\Rightarrow \pm\theta$ DOESN'T MATTER

$$\underbrace{\cos(2\pi - \theta) = \cos(-\theta)}_{\text{PERIOD } 2\pi} = \underbrace{\cos(\theta)}_{\text{EVEN}}$$

Note: IF $\vec{a} \perp \vec{b}$ (PERPENDICULAR, $\theta = \frac{\pi}{2}$)

$$\Rightarrow \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \underbrace{\cos\left(\frac{\pi}{2}\right)}_{0} = 0$$



CONVERSELY, IF $\vec{a} \neq \vec{0}$, $\vec{b} \neq \vec{0}$,

AND IF $\vec{a} \cdot \vec{b} = 0$ $\left(|\vec{a}||\vec{b}| \cos\theta = 0 \right)$

THEN $\cos\theta = 0$

$$\Rightarrow \theta = \pm \frac{\pi}{2} + n 2\pi$$

i.e. $\vec{a} \perp \vec{b}$

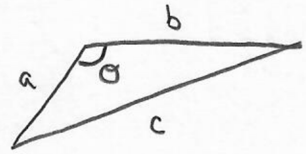
WE SAY $\vec{a}$ & $\vec{b}$ ARE <u>ORTHOGONAL</u>

(DEF)

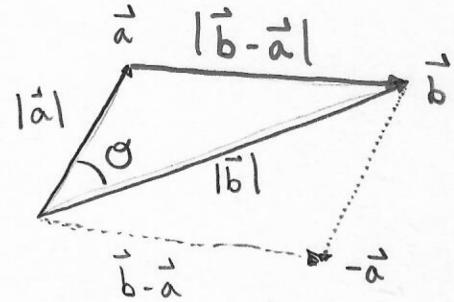
$$\boxed{\vec{a} \cdot \vec{b} = 0 \iff \vec{a} \perp \vec{b}}$$

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

LAW OF COSINES:



$$\boxed{c^2 = a^2 + b^2 - 2ab\cos\theta}$$



$$|\vec{b}-\vec{a}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$$

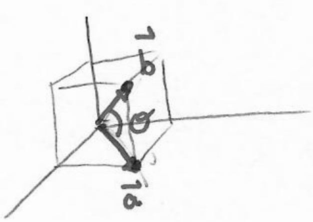
↓ PROP. (i)

$$(\vec{b}-\vec{a})\cdot(\vec{b}-\vec{a}) = \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} - 2|\vec{a}||\vec{b}|\cos\theta$$

$$\cancel{\vec{b}\cdot\vec{b}} - 2\vec{a}\cdot\vec{b} + \cancel{\vec{a}\cdot\vec{a}} = \cancel{\vec{a}\cdot\vec{a}} + \cancel{\vec{b}\cdot\vec{b}} - 2|\vec{a}||\vec{b}|\cos\theta$$

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta \quad \checkmark$$

ex. FIND THE ANGLE BETWEEN $\vec{a} = \langle 1,1,0\rangle$ & $\vec{b} = \langle 1,1,1\rangle$



GEO. DEF. OF DOT PROD.

$$\frac{\text{RAD}}{\cancel{2\pi}\ \pi} = \frac{\text{DEG}}{\cancel{360}\ 180}$$

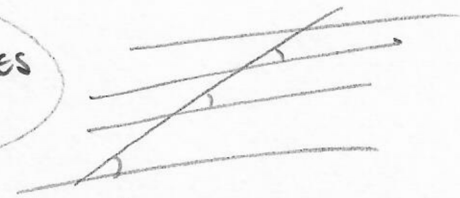
$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

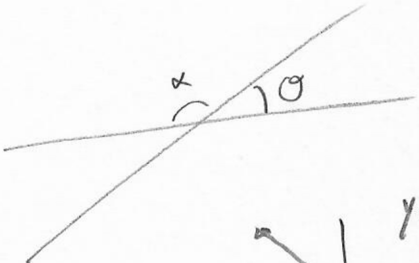
$$(1)(1) + (1)(1) + (0)(1) = \sqrt{2}\sqrt{3}\cos\theta$$

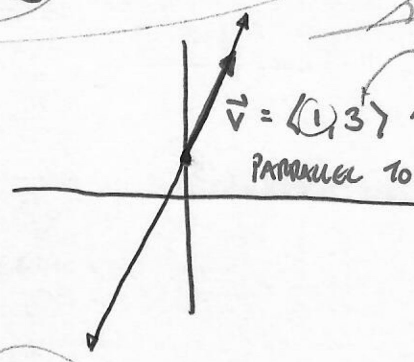
$$\frac{2}{\sqrt{6}} = \cos\theta \implies \theta = \cos^{-1}\left(\frac{2}{\sqrt{6}}\right) \approx 35^\circ$$

ex. FIND THE ACUTE ANGLE OF INTERSECTION OF THE LINES

$y = 3x + 1 \quad , \quad y = -x + 3$



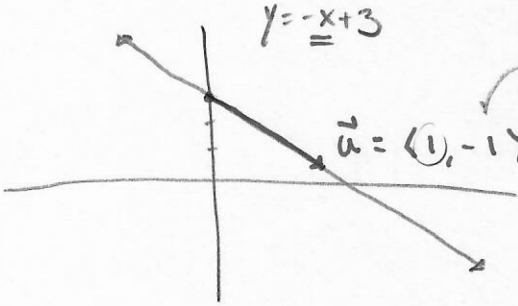




$\vec{v} = \langle 1, 3 \rangle$

PARALLEL TO $y = 3x + 1$

$y = -x + 3$

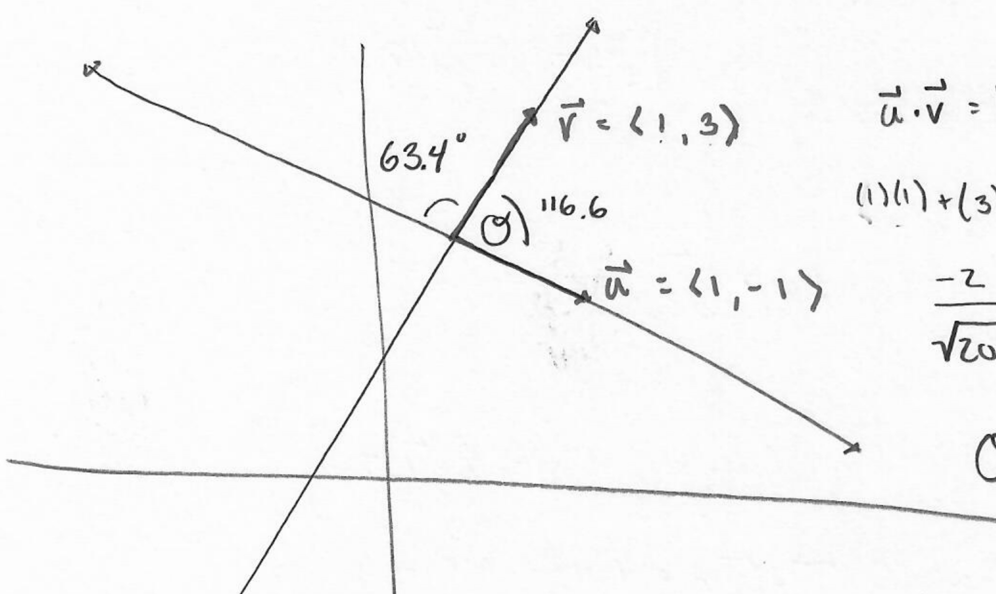


$\vec{u} = \langle 1, -1 \rangle$ PARALLEL TO

$y = -x + 3$

NOTE: IN GENERAL, $\vec{v} = \langle 1, m \rangle$ IS $\parallel$ TO THE

LINE $y = mx + b$



$\vec{v} = \langle 1, 3 \rangle$

$\vec{u} = \langle 1, -1 \rangle$

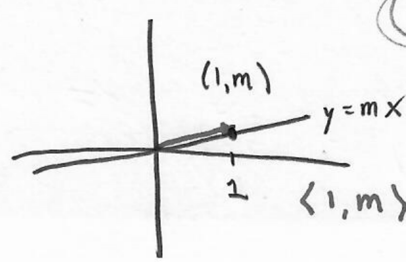
$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$

$(1)(1) + (3)(-1) = \sqrt{10}\sqrt{2} \cos\theta$

$\dfrac{-2}{\sqrt{20}} = \cos\theta$

$\theta = 116.6°$

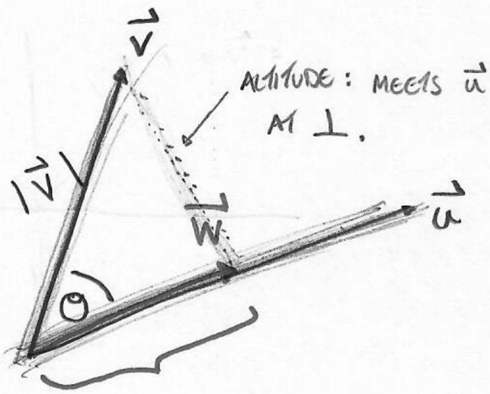
OR $63.4°$



$\langle 1, m \rangle$

$\langle 0, 1 \rangle = \hat{j}$

COMPONENTS & PROJECTIONS



THE SHADOW CAST BY $\vec{v}$ ON $\vec{u}$.

i.e. THE PROJECTION OF $\vec{v}$ ONTO $\vec{u}$.

$$\vec{w} = \text{PROJ}_{\vec{u}}\,\vec{v} \qquad (\text{VECTOR})$$

LENGTH OF THE PROJECTION

$$\text{COMP}_{\vec{u}}\,\vec{v} = \left|\text{PROJ}_{\vec{u}}\,\vec{v}\right|$$

$$= |\vec{v}|\cos\theta$$

COMPONENT OF $\vec{v}$ IN DIRECTION OF $\vec{u}$

(COMPONENT $\parallel$ TO $\vec{u}$)

$$= \left(|\vec{v}|\cos\theta\right)\frac{\vec{u}}{|\vec{u}|}$$

UNIT VECTOR IN DIRECTION OF $\vec{u}$

$$\text{COMP}_{\vec{u}}\,\vec{v} = |\vec{v}|\cos\theta = \frac{|\vec{u}||\vec{v}|\cos\theta}{|\vec{u}|}$$

$$\boxed{\text{COMP}_{\vec{u}}\,\vec{v} = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|}}$$

SCALAR

$\text{COMP}_{\vec{u}}\,\vec{v}$

$$\text{PROJ}_{\vec{u}}\,\vec{v} = \left(|\vec{v}|\cos\theta\right)\frac{\vec{u}}{|\vec{u}|}$$

$$\text{PROJ}_{\vec{u}}\,\vec{v} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}|}\right)\frac{\vec{u}}{|\vec{u}|}$$

$$\boxed{\text{PROJ}_{\vec{u}}\,\vec{v} = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|^2}\,\vec{u}}$$

$\boxed{\S 11.1,\ 11.3,\ 12.1-3}$

Quiz #5