QUIZ #5 TOMORROW 8:30 (40 MIN.) PEARSON MLM

9:15 LECTURE §12.5

11.1, 11.3, 12.1-12.3

OFFICE HOURS 1:30-3:30 PM

VECTORS: $\langle x, y, z \rangle$

ADD / SUB

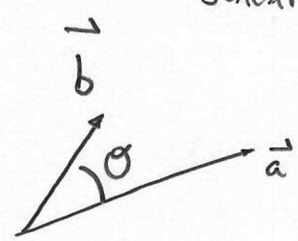
MULT BY SCALAR

2 WAYS TO MULTIPLY VECTORS

- DOT PRODUCT (SCALAR PRODUCT)

$$\langle x_1, y_1, z_1 \rangle \cdot \langle x_2, y_2, z_2 \rangle = \underbrace{x_1x_2 + y_1y_2 + z_1z_2}_{\text{SCALAR } (\mathbb{R})}$$

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| . \cos\theta$$



§12.4 CROSS PRODUCT (VECTOR PRODUCT)

↳ NOTE: CROSS PRODUCTS CAN ONLY BE TAKEN WITH 3-D VECTORS.

$$\boxed{\begin{aligned}\underline{\text{Def}}:\quad &\langle x_1, y_1, z_1 \rangle \times \langle x_2, y_2, z_2 \rangle \\ &= \langle y_1 z_2 - z_1 y_2 ,\ z_1 x_2 - x_1 z_2 ,\ x_1 y_2 - y_1 x_2 \rangle\end{aligned}}$$

TRICK FOR REMEMBERING

$$\langle x_1, y_1, z_1 \rangle \times \langle x_2, y_2, z_2 \rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

↙ ALTERNATING SUM.

$$= \hat{i} \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} - \hat{j} \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} + \hat{k} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

$$= \hat{i}\,(y_1 z_2 - z_1 y_2) - \hat{j}\,(x_1 z_2 - z_1 x_2) + \hat{k}\,(x_1 y_2 - y_1 x_2)$$

$$= \langle y_1 z_2 - z_1 y_2 ,\ z_1 x_2 - x_1 z_2 ,\ x_1 y_2 \ldots$$

e.g. $\langle 1, 0, -2\rangle \times \langle 0, 4, -1\rangle$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2 \\ 0 & 4 & -1 \end{vmatrix} = \hat{i}\begin{vmatrix} 0 & -2 \\ 4 & -1 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & -2 \\ 0 & -1 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 0 \\ 0 & 4 \end{vmatrix}$$

$$= \hat{i}\left(0-(-8)\right) - \hat{j}\left(-1-0\right) + \hat{k}\left(4-0\right)$$

$$= \langle 8, 1, 4\rangle$$

e.g. $\overbrace{\langle -3, 1, 4\rangle}^{\vec{a}} \times \overbrace{\langle 5, 0, 2\rangle}^{\vec{b}}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 1 & 4 \\ 5 & 0 & 2 \end{vmatrix} = \hat{i}\begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} - \hat{j}\begin{vmatrix} -3 & 4 \\ 5 & 2 \end{vmatrix} + \hat{k}\begin{vmatrix} -3 & 1 \\ 5 & 0 \end{vmatrix}$$

$$= \langle 2-0, 20+6, 0-5\rangle$$

$$= \underbrace{\langle 2, 26, -5\rangle}_{\vec{a}\times\vec{b}}$$

Note:

$$(\vec{a}\times\vec{b})\cdot\vec{a} = \langle 2, 26, -5\rangle \cdot \langle -3, 1, 4\rangle$$

$$= -6 + 26 - 20 = 0$$

$$(\vec{a}\times\vec{b})\cdot\vec{b} = \langle 2, 26, -5\rangle \cdot \langle 5, 0, 2\rangle$$

$$= 10 + 0 - 10 = 0$$

THM: $\vec{a} \times \vec{b}$ (THE VECTOR) IS ORTHOGONAL TO BOTH VECTOR $\vec{a}$ AND $\vec{b}$.

$\vec{u}$ & $\vec{v}$ ARE ORTHOGONAL IF $\vec{u} \cdot \vec{v} = 0$.

$$(\vec{a} \times \vec{b}) \cdot \vec{a} = (\vec{a} \times \vec{b}) \cdot \vec{b} = 0$$

ALWAYS.

RECALL: $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$

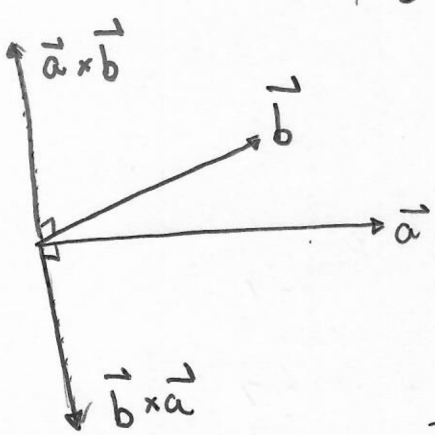
IF THIS 0 THEN $\theta = \pm \frac{\pi}{2}$ ($\vec{a}$ & $\vec{b}$ ARE PERPENDICULAR)

(AND $\vec{a} \neq 0$, $\vec{b} \neq 0$)

GEOMETRIC DEF

GIVEN $\vec{a}$ & $\vec{b}$, THE VECTOR $\vec{a} \times b$ IS $\perp$ TO BOTH $\vec{a}$ & $\vec{b}$, ITS DIRECTION IS DETERMINED BY THE RIGH HAND RULE:

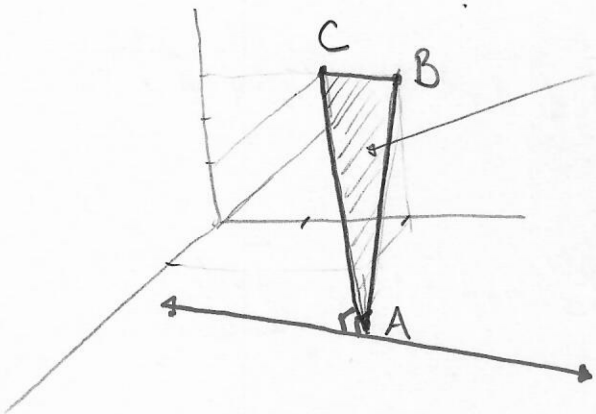
DIRECTION



- FINGERS CURL FROM $\vec{a}$ TO $\vec{b}$, THUMB POINTS IN DIRECTION OF $\vec{a} \times \vec{b}$.

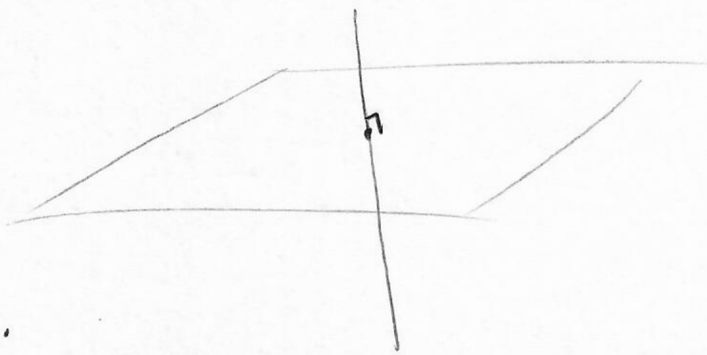
- NOTE: $\vec{b} \times \vec{a}$ POINTS IN THE OPPOSITE DIRECTION FROM $\vec{a} \times \vec{b}$

ex. Find a vector perpendicular to the plane containing the points $A(1, 2, -1)$, $B(0, 2, 3)$, $C(-2, 0, 1)$.



This Δ is contained in the plane containing A, B, C.

(3 points determine a plane)



Note: The vector $\perp$ to the plane must be $\perp$ to all vectors in the plane.

i.e. $\overrightarrow{AB}$ & $\overrightarrow{AC}$.

Take $\overrightarrow{AB} \times \overrightarrow{AC} = \langle 0-1, 2-2, 3-(-1) \rangle$

$\times \langle -2-1, 0-2, 1-(-1) \rangle$

$\langle -1, 0, 4 \rangle \times \langle -3, -2, 2 \rangle$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & 4 \\ -3 & -2 & 2 \end{vmatrix} = \langle 8, -10, 2 \rangle$$

$\perp$ to plane

and so is

$c \langle 8, -10, 2 \rangle$ for any $c \neq 0$.

MAGNITUDE:

IF θ IS THE ANGLE BETWEEN $\vec{a}$ & $\vec{b}$ $(0 \le \theta \le \pi)$ THEN $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$

$\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$

PROOF:

$$|\vec{a} \times \vec{b}|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$$

$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$

$$= |\vec{a}|^2|\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$$

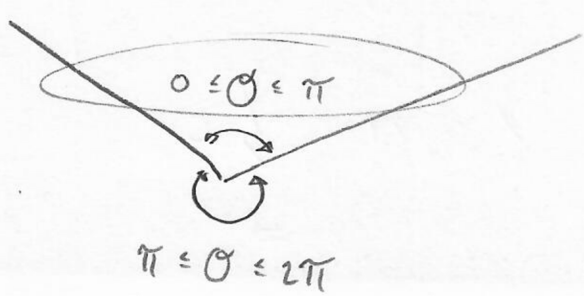
$$= |\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2\cos^2\theta$$

$$= |\vec{a}|^2|\vec{b}|(1 - \cos^2\theta) = |\vec{a}|^2|\vec{b}|^2\sin^2\theta$$

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta \quad \checkmark$$

NOTE: $0 \le \sin\theta \le 1$

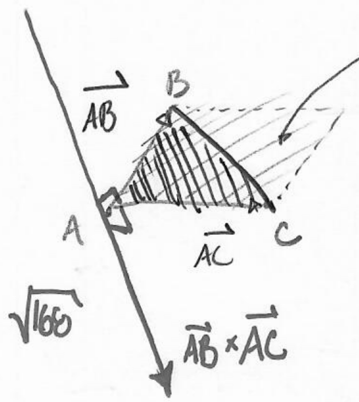
$(0 \le \theta \le \pi)$



ex. FIND THE AREA OF THE TRIANGLE ΔABC

$A(1,2,-1)$, $B(0,2,3)$, $C(-2,0,1)$

$\overrightarrow{AB} = \langle -1, 0, 4\rangle$, $\overrightarrow{AC} = \langle -3, -2, 2\rangle$

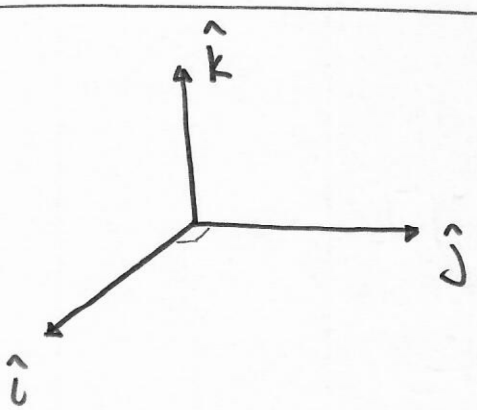


$$\text{AREA } \square = |\overrightarrow{AB} \times \overrightarrow{AC}| = |\langle 8, -10, 2\rangle|$$

$$= \sqrt{8^2 + (-10)^2 + 2^2}$$

$$= \sqrt{64 + 100 + 4} = \sqrt{168}$$

$$\text{AREA } \Delta ABC = \frac{1}{2} \text{ AREA } \square = \frac{1}{2}\sqrt{168}$$



·) $\hat{i} \times \hat{j} = \hat{k}$

• MAGNITUDE $|\hat{i} \times \hat{j}| = |\hat{i}||\hat{j}| \sin\theta$

$= (1)(1) \sin \frac{\pi}{2}$

$= 1$

• DIRECTION : RIGHT HAND RULE

DIR. $+z$ - AXIS

·) $\hat{j} \times \hat{k} = \hat{i}$

·) $\hat{k} \times \hat{i} = \hat{j}$

·) $\hat{i} \times \hat{k} = -\hat{j}$

$\left.\begin{matrix} \hat{k} \times \hat{i} = \hat{j} \\ \hat{i} \times \hat{k} = -\hat{j} \end{matrix}\right\}$ ANTI - COMMUTATIVE.

$\vec{a} \cdot \vec{b}$ SCALAR

$\vec{a} \times \vec{b}$ VECTOR

$\vec{a} \cdot (\underbrace{\vec{b} \times \vec{c}}_{\text{VECTOR}}) =$ SCALAR

VECTORS/SCALARS

$\vec{a} \times (\vec{b} \times c) =$ VECTOR

$\vec{a} \times (\underbrace{\vec{b} \cdot \vec{c}}) =$ NON-SENSE

$\underbrace{\text{VECTOR} \times \text{SCALAR}}_{?}$

$(\underbrace{\vec{a} \cdot \vec{b}}_{\text{SCALAR}})\vec{c} =$ VECTOR

$\underbrace{\vec{u}\vec{v}}$ NON-SENSE

WHICH TYPE OF MULT.

DOT /SCALAR

$\overbrace{(\vec{a} \cdot \vec{b})}^{\text{SCALAR}} \times \overbrace{(\vec{c} \cdot \vec{d})}^{\text{SCALAR}} =$ NON-SENSE

$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) =$ SCALAR.

Properties of the Cross Product

Let $\vec{a}, \vec{b}, \vec{c}$ be vectors, let c be scalar

(i) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

(ii) $(c\vec{a}) \times \vec{b} = \vec{a} \times (c\vec{b}) = c(\vec{a} \times \vec{b})$

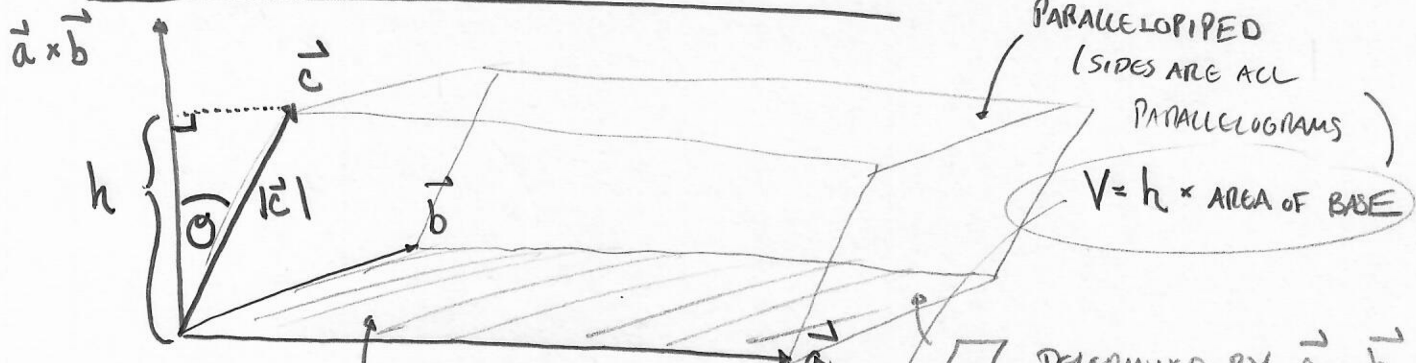
(iii) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$ (Left - Distribution)

(iv) $(\vec{a} + \vec{b}) \times \vec{c} = (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c})$

(v) $\underbrace{\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{c} \times \vec{a})}_{\text{Sacar Triple Product.}}$

Volume of Parallelopiped determined by $\vec{a}, \vec{b}, \vec{c}$.

Scalar Triple Product (Geometric Def)



▱ determined by $\vec{a}, \vec{b}$

Area $= |\vec{a} \times \vec{b}|$

Area of Base $= |\vec{a} \times \vec{b}|$

$h = |\vec{c}| \cos\theta$

$V = (|\vec{c}| \cos\theta) |\vec{a} \times \vec{b}|$

$V = |\vec{c}| |\vec{a} \times \vec{b}| \cos\theta = \vec{c} \cdot (\vec{a} \times \vec{b})$ Scalar Triple Product