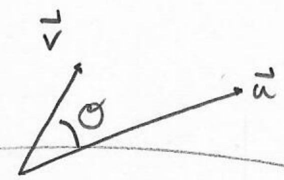
§12.5 EQ'S OF LINES & PLANES

$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$



IF $\theta = \pm \frac{\pi}{2}$ THEN $\vec{u} \cdot \vec{v} = 0$ PERPINDICULAR VECTORS

IF $\vec{u} \cdot \vec{v} = 0$ THEN $\theta = \pm \frac{\pi}{2}$

$(\vec{u} = \vec{0}, \vec{v} = \vec{0})$

ORTHOGONAL VECTORS

SAME

$\perp$ PERPINDICULAR : MEET AT RIGHT ANGLE.

ORTHOGONAL : $\vec{u} \cdot \vec{v} = 0$; $x_1 x_2 + y_1 y_2 + z_1 z_2 = 0$.

n-DIMENSIONAL VECTORS: $(n \geq 1)$

4D: $\vec{u} = \langle 2, 3, 5, -7 \rangle$

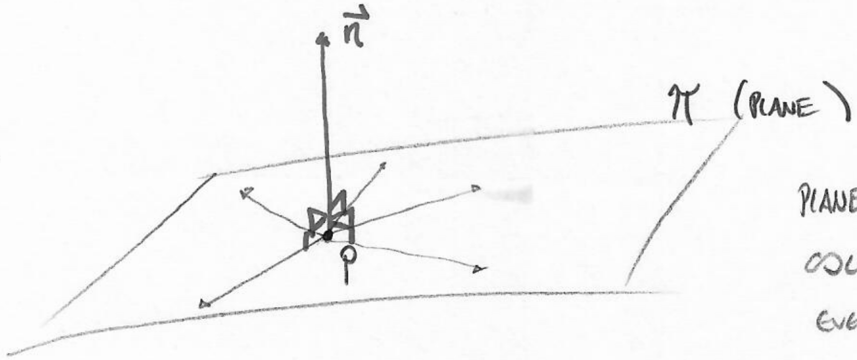
$\vec{v} = \langle 5, 3, -1, 2 \rangle$

$\vec{u}$ & $\vec{v}$ ARE $\perp$

$\Updownarrow$

$\vec{u} \cdot \vec{v} = 10 + 9 - 5 - 14 = 0$ ✓

PLANES



PLANE EXTENDS ONLY FAR IN EVERY DIRECTION

Def: A VECTOR $\vec{n}$ IS NORMAL TO THE PLANE π IF IT IS $\perp$ TO EVERY VECTOR "IN THE PLANE"

(PARALLEL TO THE PLANE.)

(i.e. PERPINDICULAR TO THE PLANE)

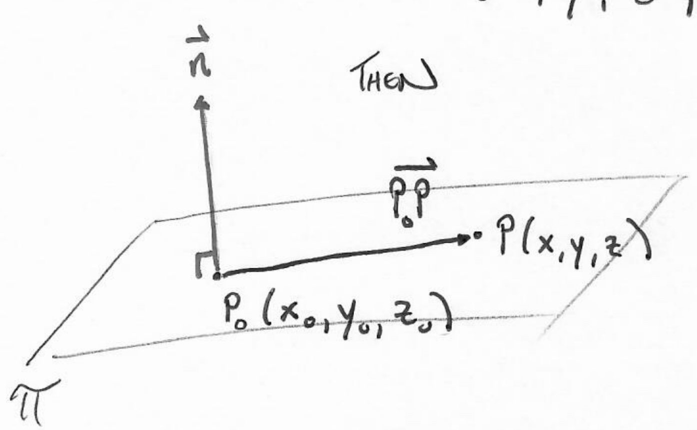
(A.K.A. $\vec{n}$ IS A NORMAL VECTOR OF THE PLANE π.)

EQ OF A PLANE.

LET $P_0(x_0, y_0, z_0)$ BE A POINT IN THE PLANE π, AND LET $\vec{n}$ BE A NORMAL VECTOR OF THE PLANE π. $\vec{n} = \langle a, b, c \rangle$

$\langle a, b, c \rangle$

IF $P(x, y, z)$ IS ANY OTHER POINT IN THE PLANE, THEN



$$\vec{n} \cdot \overrightarrow{P_0P} = 0$$

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz \underbrace{- ax_0 - by_0 - cz_0} = 0$$

EQ OF THE PLANE π : $\boxed{ax + by + cz = d}$ *

WHERE $\vec{n} = \langle a, b, c \rangle$ & $d = ax_0 + by_0 + cy_0$

$= \vec{n} \cdot \langle x_0, y_0, z_0 \rangle$

ex. EQ: $2x - 7y + 4z = 4$ ← PLANE!

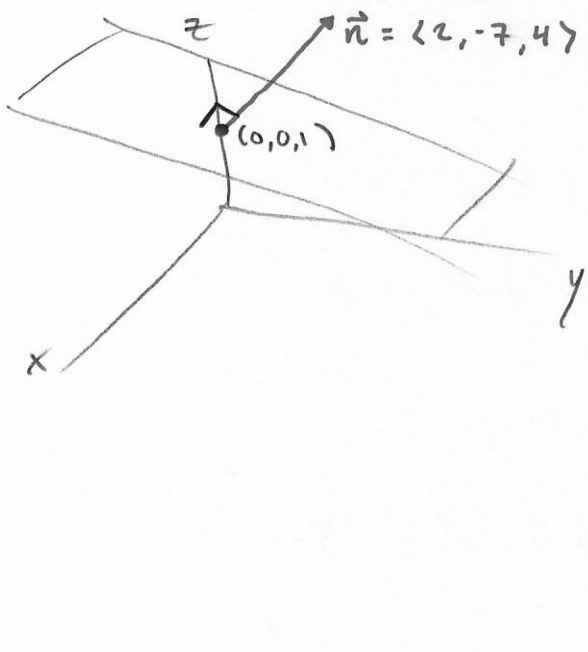
THIS PLANE HAS NORMAL VECTOR $\vec{n} = \langle 2, -7, 4 \rangle$

NOTE* NORMAL VECTORS ARE NOT UNIQUE:

$\underbrace{c\langle 2, -7, 4 \rangle}$, $c \neq 0$

NORMAL TO π FOR EVERY SCALAR $c \neq 0$.

POINT IN THE PLANE : $(0, 0, 1)$ (NOT UNIQUE!)



ex. FIND AN EQUATION FOR THE PLANE THROUGH

$(5, -2, 1)$ WITH NORMAL VECTOR $\vec{n} = \langle 2, -3, 1 \rangle$.

$$\begin{array}{ccc} & & \\ a & b & c \end{array}$$

$$\boxed{ax + by + cz = d \quad , \quad \vec{n} = \langle a, b, c \rangle}$$

TEMPLATE

$2x - 3y + z = d$ FIND d.

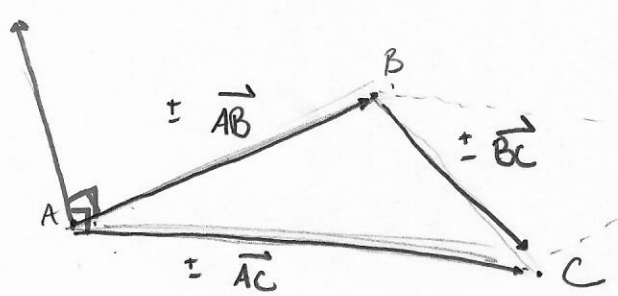
$\overset{x}{(5}, \overset{y}{-2}, \overset{z}{1})$ MUST SATISFY THIS EQ.

$$\underset{10}{2(5)} \underset{+6}{- 3(-2)} \underset{+1}{+ (1)} = \underline{d = 17}$$

$$\boxed{2x - 3y + z = 17}$$

ex. FIND AN EQ. OF THE PLANE THROUGH

$\underline{A(1, 0, 3)}$, $B(0, 5, -2)$, $C(-4, -1, 0)$



WE NEED:

(1) POINT IN PLANE

(2) $\vec{n}$ NORMAL VECTOR OF THE PLANE.

RECALL: $\overrightarrow{AB} \times \overrightarrow{AC}$ = A VECTOR $\perp$ TO BOTH $\overrightarrow{AB}$ & $\overrightarrow{AC}$

$\overrightarrow{AB} = \langle -1, 5, -5 \rangle$

$\overrightarrow{AC} = \langle -5, -1, -3 \rangle$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 5 & -5 \\ -5 & -1 & -3 \end{vmatrix} = \langle -15-5,\ 25-3,\ 1+25 \rangle$$

$$= \langle -20, 22, 26 \rangle$$

or $\langle -10, 11, 13 \rangle$ SAME DIRECTION

$\quad\ \ a \quad b \quad c$

$$ax + by + cz = d$$

$$-10x + 11y + 13z = d$$

SOL'N/ POINT : $(1, 0, 3)$

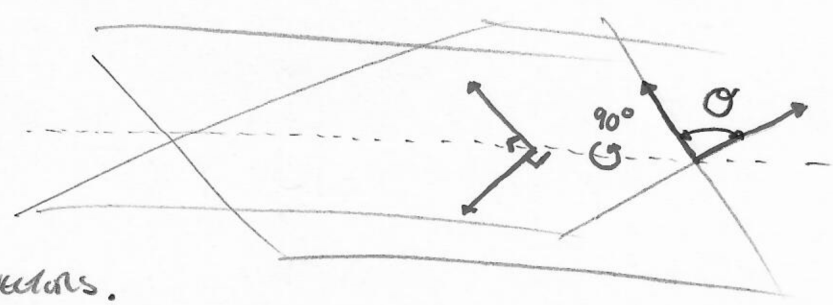
$$-10(1) + 11(0) + 13(3) = d = 29$$

$$-10 + 0 + 39$$

$$\boxed{-10x + 11y + 13z = 29}$$

ex. FIND THE ANGLE BETWEEN THE PLANES

$$2x + 2y + 2z = 3$$
$$2x - 2y - z = 5$$



= ANGLE BETWEEN THE NORMAL VECTORS.

$$\vec{n}_1 = \langle 2, 2, 2 \rangle, \quad \vec{n}_2 = \langle 2, -2, -1 \rangle$$

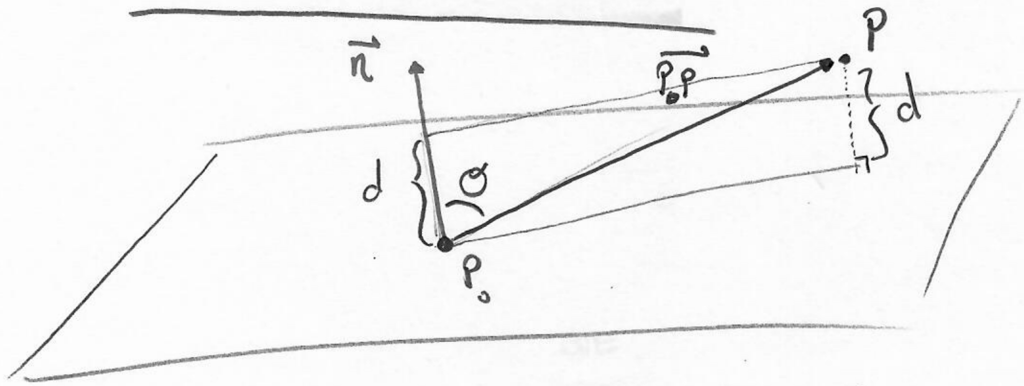
$$\vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1||\vec{n}_2| \cos\theta$$

$$4 - 4 - 2 = \sqrt{2^2 + 2^2 + 2^2}\sqrt{2^2 + 2^2 + 1^2} \cos\theta$$

$$-2 = \sqrt{12}\sqrt{9} \cos\theta \implies \cos\theta = \frac{-2}{6\sqrt{3}}$$

$$\theta = \cos^{-1}\left(\frac{-2}{6\sqrt{3}}\right)$$

DISTANCE BETWEEN A POINT & A PLANE.

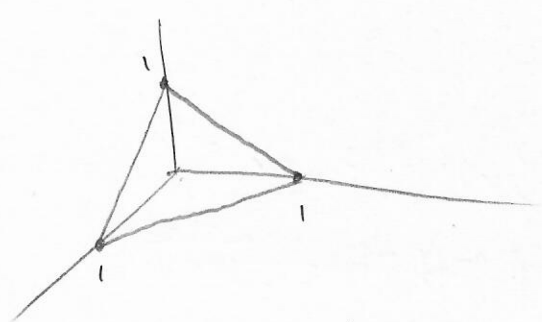


$$\cos\theta = \frac{d}{|\overrightarrow{P_0P}|} \Rightarrow d = |\overrightarrow{P_0P}|\cos\theta$$

$$d = \frac{|\overrightarrow{P_0P}||\vec{n}|\cos\theta}{|\vec{n}|}$$

$$\boxed{d = \frac{|\overrightarrow{P_0P}\cdot\vec{n}|}{|\vec{n}|} = \text{Comp}_{\vec{n}}\,\overrightarrow{P_0P}}$$

ex. FIND DISTANCE FROM THE ORIGIN TO THE PLANE $x+y+z=1$.



POINT IN PLANE $(1,0,0)$ P_0

$\vec{n} = \langle 1,1,1\rangle$

POINT $P(0,0,0)$

$$d = \frac{|\overrightarrow{P_0P}\cdot\vec{n}|}{|\vec{n}|} = \frac{|\langle -1,0,0\rangle\cdot\langle 1,1,1\rangle|}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

TIPS:

LET $\vec{u}, \vec{v}$ BE VECTORS, π_1 & π_2 BE PLANES

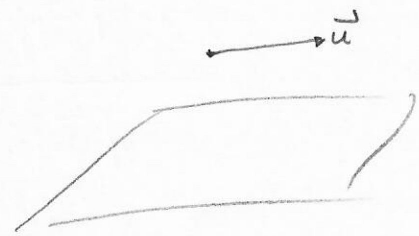
WITH NORMAL VECTORS $\vec{n_1}$ & $\vec{n_2}$, RESPECTIVELY.

2 VECTORS

(1) $\vec{u} \parallel \vec{v} \iff \vec{u} = c\vec{v}$, $c \in \mathbb{R}$, $c \neq 0$

PARALLEL

(2) $\vec{u} \perp \vec{v} \iff \vec{u} \cdot \vec{v} = 0$



1 VECTOR
1 PLANE

(3) $\vec{u} \parallel \pi_1 \iff \vec{u} \cdot \vec{n_1} = 0$

"PARALLEL TO PLANE" $\iff$ "PERPINDICULAR TO NORMAL"

(4) $\vec{u} \perp \pi_1 \iff \vec{u}_1 = c\vec{v}$, $c \in \mathbb{R}$, $c \neq 0$

"PERPINDICULAR TO PLANE" $\iff$ "PARALLEL TO NORMAL VECTOR"

(5) $\pi_1 \parallel \pi_2 \iff \vec{n_1} \parallel \vec{n_2}$

(6) $\pi_1 \perp \pi_2 \iff \vec{n_1} \perp \vec{n_2}$

PAY SPECIAL ATTENTION TO § 12.5 HW

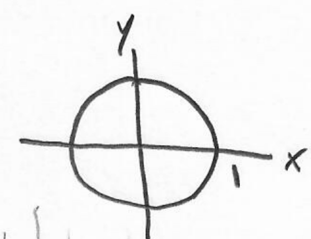
§12.6 CYLINDERS & QUADRIC SURFACES

Def: THE GRAPH OF AN EQ IN 3 VARIABLES IS THE SET OF ALL POINTS (x, y, z) THAT SATISFY THE EQUATION.

WE WANT TO BE ABLE TO DESCRIBE THE GRAPHS OF CERTAIN EQUATIONS.

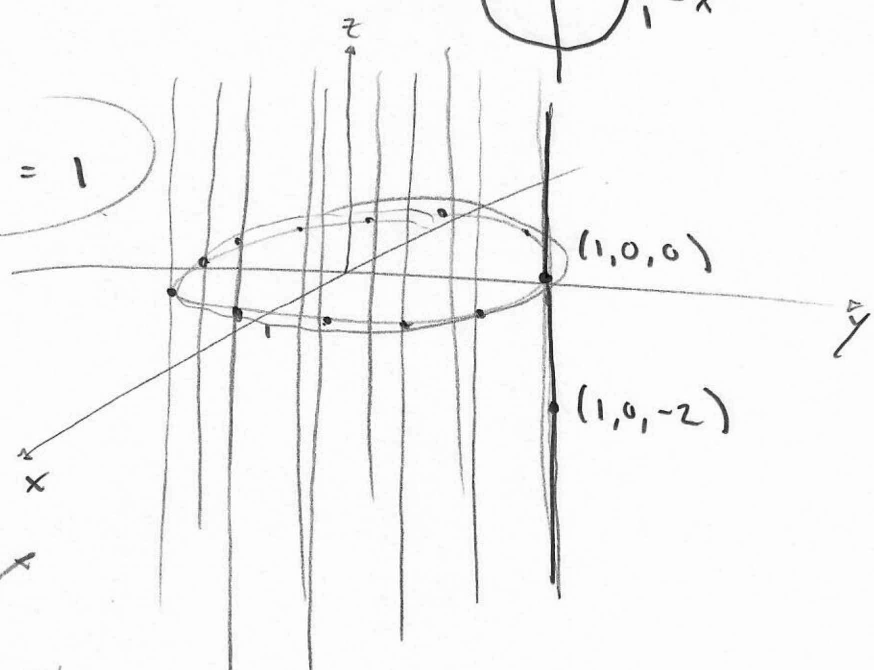
CYLINDERS

IN 2D $x^2 + y^2 = 1$



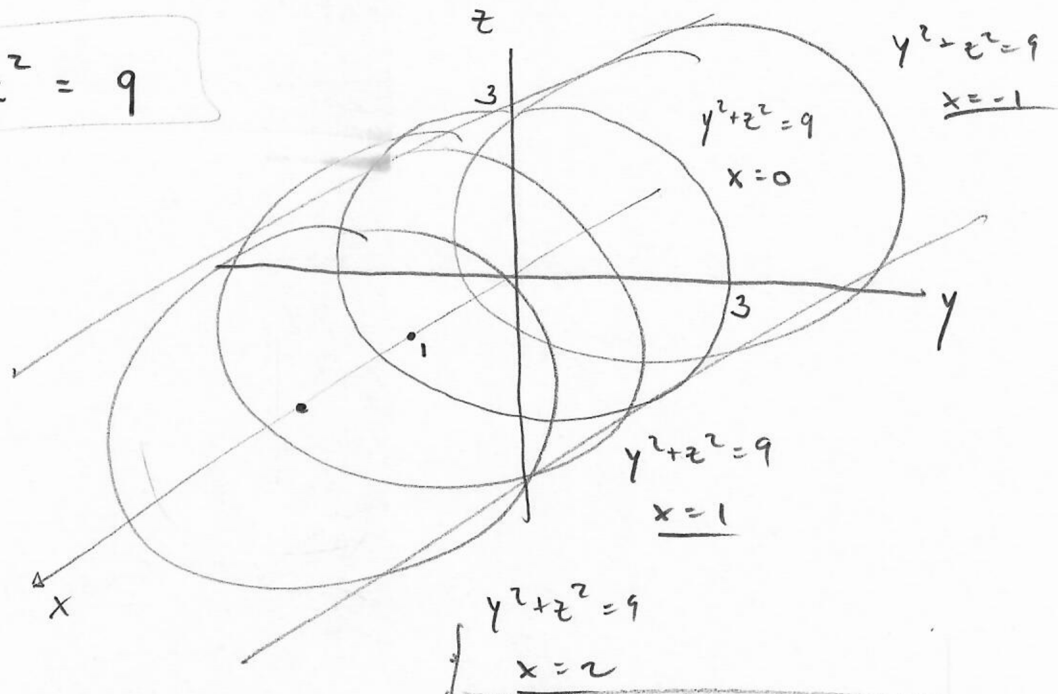
IN 3D $x^2 + y^2 = 1$

No Z-TERM



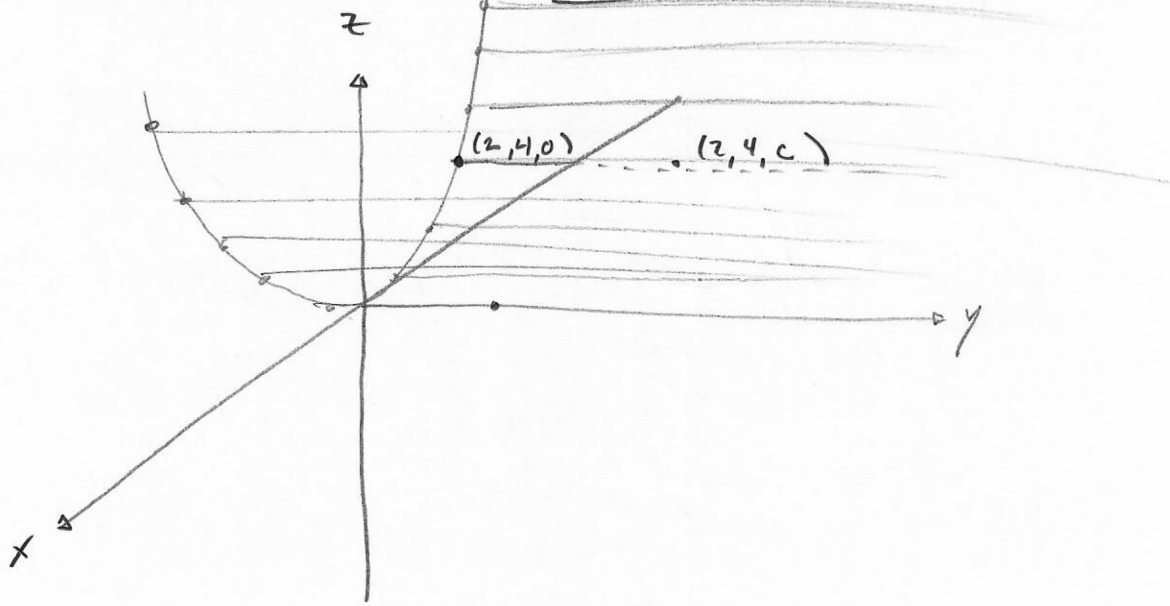
THROUGH EVERY POINT SATISFYING THE EQUATION $x^2 + y^2 = 1$, EVERY POINT ON THE LINE THROUGH THAT POINTS // TO Z-AXIS ALSO SATISFIES THE EQUATION.

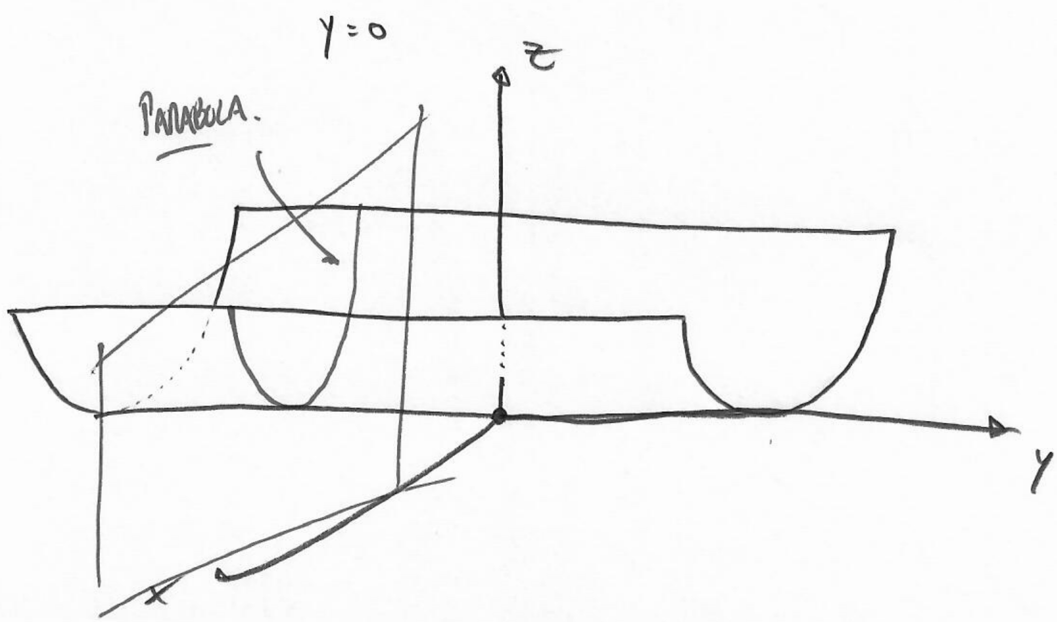
ex. $\boxed{y^2 + z^2 = 9}$



SAME SHAPE (CIRCLE) IN DIFFERENT PLANES $x = c$

ex. $z = x^2$

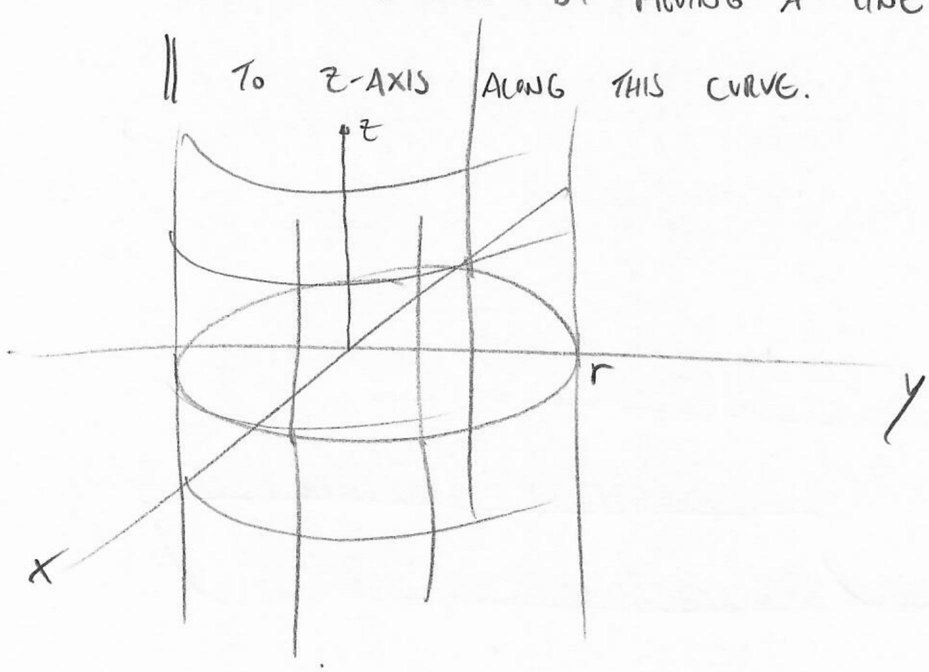




Def: A CYLINDER IS A SURFACE GENERATED BY MOVING A STRAIGHT LINE ALONG A GIVEN PLANAR (FLAT) CURVE WHILE HOLDING THE LINE $\parallel$ TO A GIVEN FIXED LINE.

IN PARTICULAR: $x^2 + y^2 = r^2$ (PLANE CURVE)

GRAPH IS OBTAINED BY MOVING A LINE $\parallel$ TO Z-AXIS ALONG THIS CURVE.



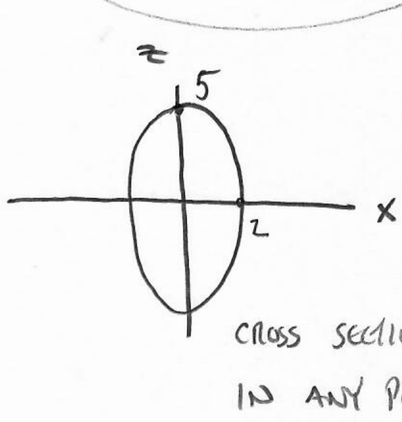
☆ GIVEN AN EQUATION WITH ONLY 2 VARIABLES

(e.g. y & z) THEN

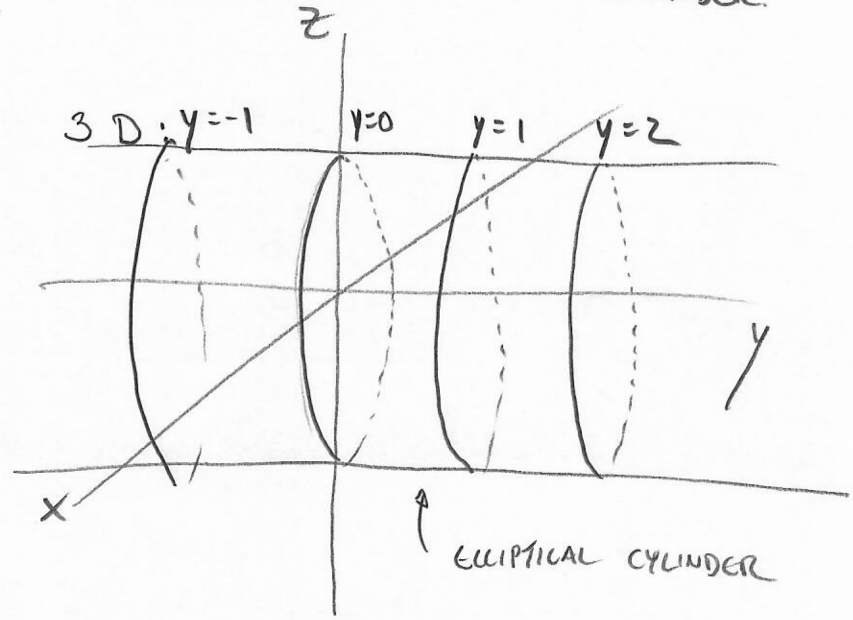
1. SKETCH THE GRAPH IN 2 DIM
   (y & z AXIS ONLY)

2. THE GRAPH OF THE EQ IN 3 DIM IS THE CYLINDER GENERATED BY MOVING A LINE $\parallel$ TO THE MISSING VARIABLE'S AXIS ALONG THE CURVE FROM STEP 1.

ex. $\dfrac{x^2}{4} + \dfrac{z^2}{25} = 1$ ← ONLY 2 VAR ⇒ CYLINDER



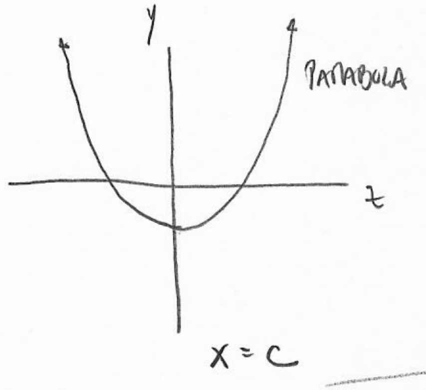
CROSS SECTION IN ANY PLANE $y = C$

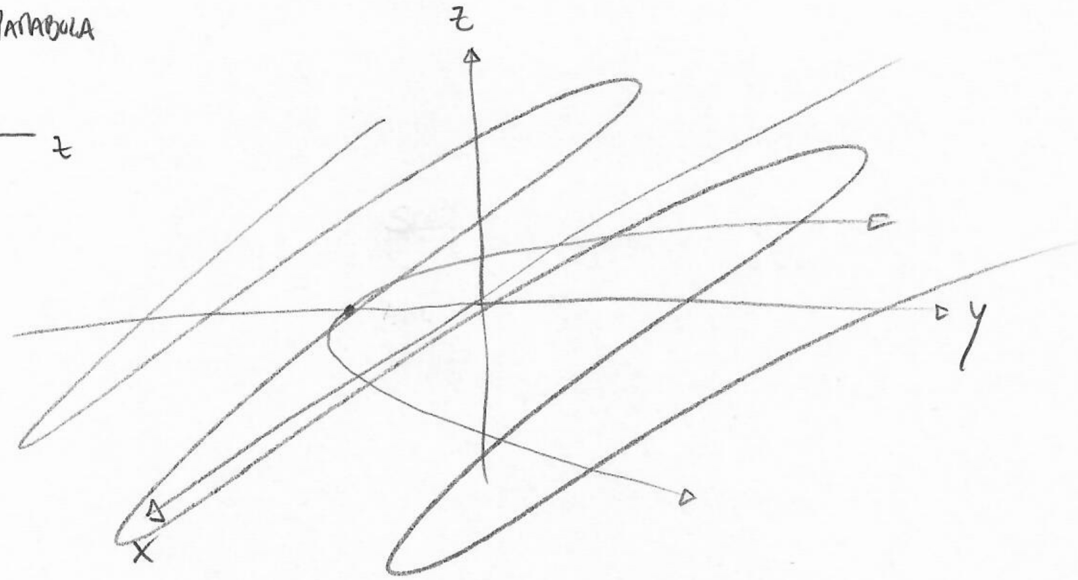


ELLIPTICAL CYLINDER

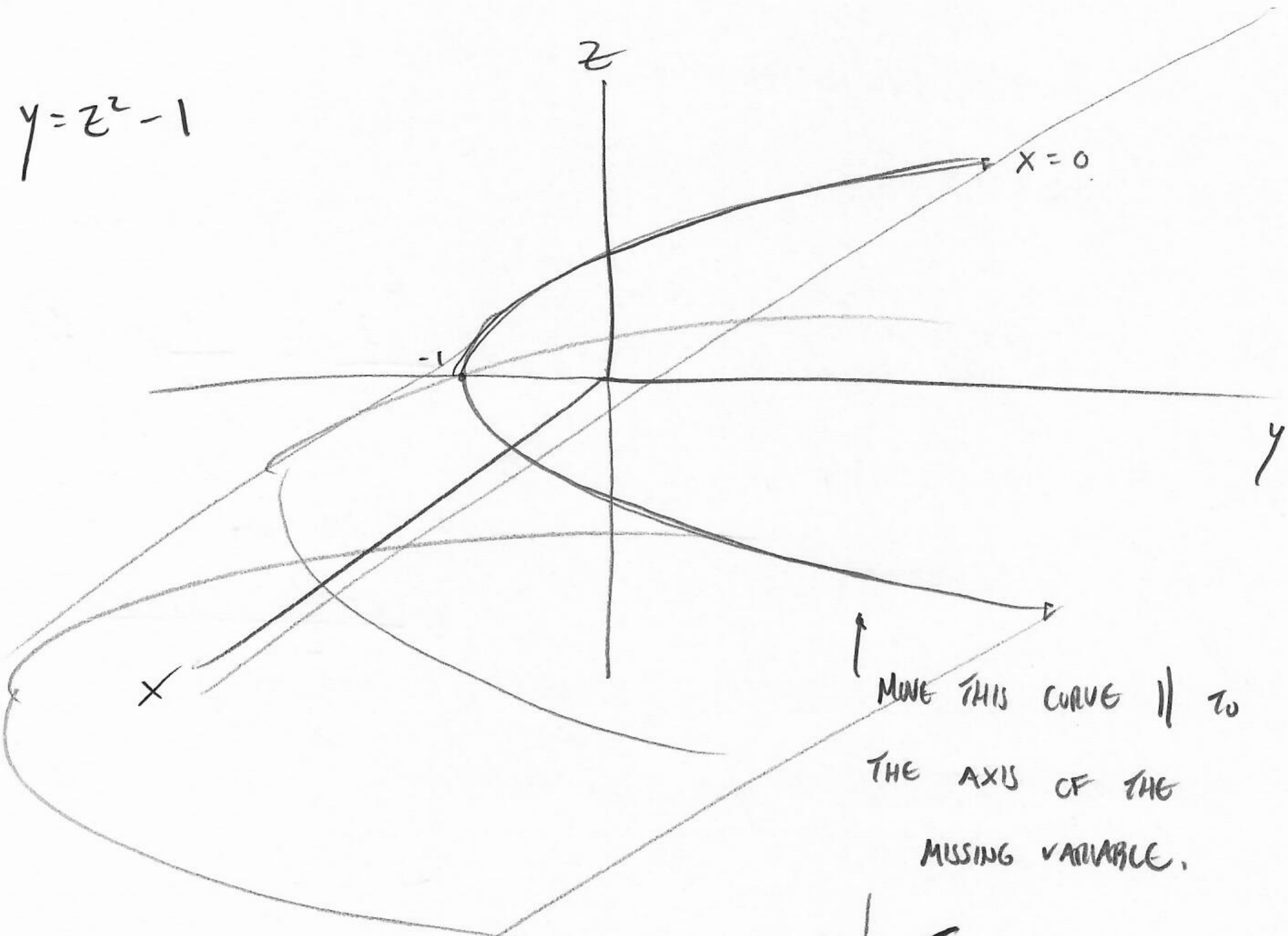
ex. $y = z^2 - 1$

SKETCH THIS GRAPH IN 3D AND DESCRIBE THE GRAPH.





$y = z^2 - 1$



MOVE THIS CURVE || TO THE AXIS OF THE MISSING VARIABLE.