§ 10.7 VECTOR FUNCTIONS & SPACE CURVES

FOR $I \subset \mathbb{R}$, $\vec{r} : I \to \mathbb{R}^n$ IS CALLED A VECTOR-VALUED FUNCTION. WE FOCUS ON THE CASE $n = 3$.

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$
$$= \underbrace{f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}}$$

$f, g, h : I \to \mathbb{R}$ REAL-VALUED COMPONENT FUNCTIONS

e.g. $\vec{r}(t) = \langle \sqrt{t+4}, \ln(1-t), \frac{1}{t} \rangle$

DOMAIN: $t + 4 \geq 0$ $\quad$ $1 - t > 0$ $\quad$ $t \neq 0$

$t \geq -4$ $\quad$ $1 > t$

$$[-4, 0) \cup (0, 1)$$

Def: IF $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

THEN $\lim_{t \to a} = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle$

PROVIDED THE LIMITS OF COMP. FUNC'S EXIST.

Def: A VECTOR FUNCTION $\vec{r}$ IS CONTINUOUS AT a IF

$$\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$$

(THAT IS, IFF COMPONENT FUNC'S CONT. AT a)

NOW SUPPOSE $I \subset \mathbb{R}$ IS AN INTERVAL,

f, g, h ARE CONTINUOUS ON I

AND $\vec{r} : I \to \mathbb{R}^3$

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

THEN $C = \{(x, y, z) : x = f(t),\ y = g(t),\ z = h(t),\ t \in I\}$ IS CALLED A SPACE CURVE & f, g, h ARE CALLED PARAMETRIC EQ'S OF C, t IS CALLED PARAMETER.

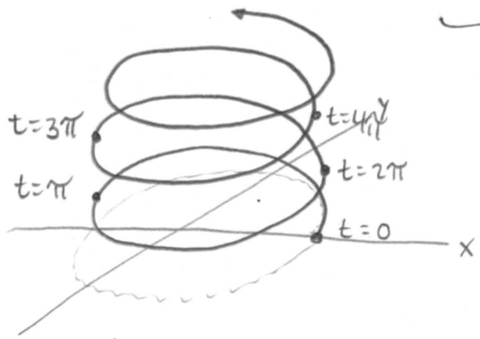
C IS TRACED OUT BY PARTICLE WHOSE POSITION AT TIME t IS GIVEN BY $\vec{r}(t)$.

e.g. $\vec{r}(t) = \langle x_0 + at,\ y_0 + bt,\ z_0 + ct \rangle$

$$= \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$$

LINE THROUGH $\parallel$ TO

e.g. $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$



"HELIX"

- CURVE C LIES ON CYLINDER $x^2 + y^2 = 1$
- PROJECTION OF CURVE ONTO x-y PLANE IS $\vec{r}(t) = \langle \cos t, \sin t, 0 \rangle$

C.C. CIRCLE

ex. FIND EQ OF LINE SEGMENT THAT JOINS $P(2,4,-3)$ TO $Q(5,-6,1)$

WE NEED $\vec{r}(0) = \langle 2,4,-3 \rangle$

$\vec{r}(1) = \langle 5,-6,1 \rangle$

& COMPONENTS SHOULD CHANGE AT CONSTANT RATE

$$\vec{r}(t) = (1-t)\langle 2,4,-3 \rangle + t\langle 5,-6,1 \rangle$$

$$\left(\vec{r}(t) = (1+t)P + tQ \right)$$

ex. FIND VECTOR FUNCTION THAT REPRESENTS CURVE OF INTERSECTION OF CYLINDER $x^2 + y^2 = 1$ AND PLANE $y + z = 2$

LET $x = \cos t$

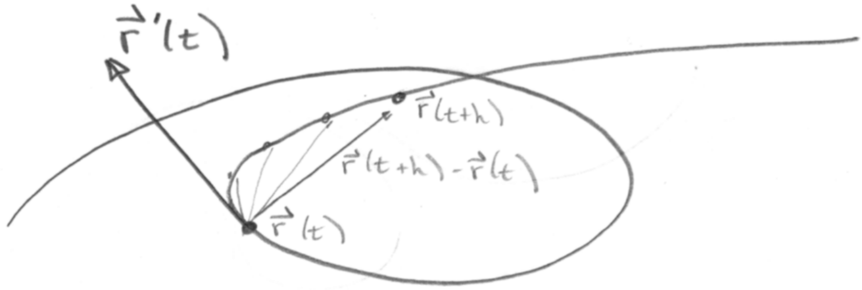
$y = \sin t$

$z = 2 - \sin t$

NOTE: NOT UNIQUE. OTHER REPARAMETRIZATIONS?

DERIVATIVES:

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$



PROVIDED THE LIMIT EXISTS, $\vec{r}\,'(t)$ IS CALLED THE TANGENT VECTOR TO THE CURVE DEFINED BY $\vec{r}$ AT THE POINT $\vec{r}(t)$

THE TANGENT LINE TO C AT $P = \vec{r}(t)$ IS DEFINED TO BE THE LINE THROUGH P, $\parallel$ TO $\vec{r}\,'(t)$.

UNIT TANGENT VECTOR $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$

THM: IF $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ & f, g, h DIFFERENTIABLE, THEN $\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$.

Pf: LIMITS ARE TAKEN COMPONENT-WISE.

ex. FIND PARAMETRIC EQ'S FOR TANGENT LINE TO CURVE

$$x = 1 + 2\sqrt{t}, \quad y = t^3 - t, \quad z = t^3 + t$$

AT $(3, 0, 2)$

↑

NOTE: CORRESPONDS TO $t = 1$.

$\therefore$ TANGENT LINE TO $\vec{r}(t) = \langle 1 + 2\sqrt{t}, t^3 - t, t^3 + t \rangle$

AT $\vec{r}(1) = (3, 0, 2)$

IS $\ell(t) = \vec{r}(1) + t\vec{r}\,'(1)$.

$$\vec{r}\,'(t) = \left\langle \frac{1}{\sqrt{t}}, 3t^2 - 1, 3t^2 + 1 \right\rangle$$

$$\vec{r}\,'(1) = \langle 1, 2, 4 \rangle$$

$$\therefore \ \ell(t) = \langle 3, 0, 2 \rangle + t\langle 1, 2, 4 \rangle$$

i.e. $x = 3 + t, \quad y = 2t, \quad z = 2 + 4t$

THM SUPPOSE $\vec{u}$ & $\vec{v}$ ARE DIFFERENTIABLE VECTOR FUNCTIONS,

c IS A SCALAR & f IS REAL-VALUED FUNCTION. THEN

1. $\frac{d}{dt}\left[\vec{u}(t)+\vec{v}(t)\right] = \vec{u}'(t) + \vec{v}'(t)$

2. $\frac{d}{dt}\left[c\vec{u}(t)\right] = c\vec{u}'(t)$

3. $\frac{d}{dt}\left[f(t)\vec{u}(t)\right] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$

4. $\frac{d}{dt}\left[\vec{u}(t)\cdot\vec{v}(t)\right] = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$

5. $\frac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right] = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)$

6. $\frac{d}{dt}\left[\vec{u}(f(t))\right] = f'(t)\vec{u}'(f(t))$

PROOF OF 3, 4, 6:

LET $\vec{u}(t) = u_1(t)\hat{e}_1 + u_2(t)\hat{e}_2 + u_3(t)\hat{e}_3$

3: $\frac{d}{dt}\left[\sum_{i=1}^{3} f(t)u_i(t)\hat{e}_i\right] = \sum_{i=1}^{3}\frac{d}{dt}\left[f(t)u_i(t)\right]\hat{e}_i$

$$= \sum_{i=1}^{3}\left(f'(t)u_i(t) + f(t)u_i'(t)\right)\hat{e}_i$$

$$= \sum_{i=1}^{3} f'(t)u_i(t)\hat{e}_i + \sum_{i=1}^{3} f(t)u_i'(t)\hat{e}_i =$$

$$= f'(t)\sum u_i\hat{e}_i + f(t)\sum u_i'\hat{e}_i = f'(t)\vec{u}(t) + f(t)\vec{u}'(t) \checkmark$$

4: $\frac{d}{dt}\left[\vec{u}(t)\cdot\vec{v}(t)\right] = \frac{d}{dt}\sum_{i=1}^{3} u_i(t)v_i(t)$

$$= \sum_{i=1}^{3}\frac{d}{dt}\left[u_i(t)v_i(t)\right] = \sum_{i=1}^{3}\left(u_i'(t)v_i(t) + u_i(t)v_i'(t)\right)$$

$$= \sum_{i=1}^{3} u_i'(t)v_i(t) + \sum_{i=1}^{3} u_i(t)v_i'(t)$$

$$= \vec{u}'(t)\cdot\vec{v}(t) + u(t)\cdot\vec{v}'(t) \quad ✓$$

6: $\frac{d}{dt}\left[\sum_{i=1}^{3} u_i(f(t))\hat{e}_i\right] = \sum_{i=1}^{3}\frac{d}{dt}\left[u_i(f(t))\right]\hat{e}_i$

$$= \sum_{i=1}^{3} u_i'(f(t))\cdot f'(t)\hat{e}_i = f'(t)\vec{u}'(f(t)) \quad ✓$$

ex. SHOW THAT IF $|\vec{r}(t)| = c$ (CONSTANT) THEN $\vec{r}'(t)$ IS ORTHOGONAL TO $\vec{r}(t)$ FOR ALL t.

Note: $\vec{r}(t)\cdot\vec{r}(t) = |\vec{r}(t)|^2 = c^2$ (CONSTANT)

$$\Rightarrow \frac{d}{dt}\left[\vec{r}(t)\cdot\vec{r}(t)\right] = \frac{d}{dt}\left[c^2\right]$$

$$2\vec{r}'(t)\cdot r(t) = 0$$

$$\vec{r}'(t)\cdot r(t) = 0$$ ORTHOGONAL $\forall t$. ✓

(IF CURVE LIES ON SPHERE, THEN TANGENT VECTOR $\perp$ TO POSITION VECTOR)

DEFINITE INTEGRAL

$$\int_a^b \vec{r}(t)\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} \vec{r}(t_i^*)\,\Delta t$$

$$= \lim_{n\to\infty} \left\langle \sum_{i=1}^{n} f(t_i^*)\,\Delta t \,,\ \sum_{i=1}^{n} g(t_i^*)\Delta t \,,\ \sum_{i=1}^{n} h(t_i^*)\Delta t \right\rangle$$

$$= \left\langle \int_a^b f(t)\,dt \,,\ \int_a^b g(t)\,dt \,,\ \int_a^b h(t)\,dt \right\rangle$$

$$= \vec{R}(b) - \vec{R}(a) \ ; \text{ WHERE } \vec{R}'(t) = \vec{r}(t) .$$

REMARK: BOTH DERIVATIVES & INTEGRALS ARE CALCULATED COMPONENT-WISE.

ex. EVALUATE : $\displaystyle\int_0^1 \left(\frac{4}{1+t^2}\,\hat{\imath} + \frac{2t}{1+t^2}\,\hat{\jmath} \right) dt$

$$= 4\int_0^1 \frac{1}{1+t^2}\,dt\,\hat{\imath} + \int_0^1 \frac{2t}{1+t^2}\,dt\,\hat{\jmath}$$

$$4\tan^{-1} t \Big|_0^1 \hat{\imath} + \ln|1+t^2| \Big|_0^1 \hat{\jmath}$$

$$= \pi\,\hat{\imath} + \ln 2\,\hat{\jmath}$$