§10.8 ARCLENGTH (OMIT CURVATURE)

2D: $\underbrace{x = f(t),\ y = g(t)}$, $a \le t \le b$

PARAM EQ'S OF PLANAR CURVE C

(TRACED OUT BY PARTICLE EXACTLY ONCE AS t VARIES FROM a TO b)

THEN ARCLENGTH $L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt$

i.e. $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

3D: $x = f(t)$, $y = g(t)$, $z = h(t)$

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\, dt$$

i.e. $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

IN SUMMARY, $L = \int_a^b |\vec{F}'(t)|\, dt$

ex.

LET C BE THE CURVE OF INTERSECTION OF THE PARABOLIC CYLINDER $x^2 = 2y$ & THE SURFACE $3z = xy$. FIND THE EXACT LENGTH OF C FROM ORIGIN TO $(6, 18, 36)$.

LET $x = t$

$y = \frac{1}{2}t^2$

$z = \frac{1}{6}t^3$

$$L = \int_0^6 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$= \int_0^6 \sqrt{1 + t^2 + \frac{1}{4}t^4}\, dt$$

$$= \int_0^6 \sqrt{\left(1 + \frac{1}{2}t^2\right)^2}\, dt = \int_0^6 1 + \frac{1}{2}t^2\, dt = t + \frac{1}{6}t^3 \Big|_0^6$$

$$= 6 + 36 = 42$$

ARC LENGTH FUNCTION $s(t) = \int_a^t |\vec{r}\,'(u)|\, du$

$\left(s'(t) = |\vec{r}\,'(t)| \quad \text{BY FTC} \right)$

NOTE: AS LONG AS $\vec{r}\,'(t) \neq 0$ ON ANY INTERVAL, S IS A STRICTLY INCREASING FUNCTION OF t. $\Rightarrow$ 1-1, i.e. INVERTIBLE.

IF WE SOLVE FOR t IN TERMS OF S, WE CAN REPARAMETRIZE A CURVE $r(t)$ AS $\vec{r}(t(s))$. (PARAMETRIZED BY ARCLENGTH).

ex. REPARAMETRIZE THE CURVE $\vec{r}(t) = \langle \cancel{e^{2t}}^{\,e^t} \cos t, 2, \cancel{e^{2t}}^{\,e^t} \sin t \rangle$

WITH RESPECT TO ARC LENGTH MEASURED FROM THE POINT $\underbrace{(1, 2, 0)}_{t=0}$

IN THE DIRECTION OF INCREASING t.

WE HAVE $$s = s(t) = \int_0^t |\vec{r}\,'(u)| \, du$$

$$= \int_0^t \sqrt{(2e^{2u}\cos u - e^{2u}\sin u)^2 + 0^2 + (2e^{2u}\sin u + e^{2u}\cos u)^2} \, du$$

$$= \int_0^t \sqrt{4e^{4u}(\cos^2 u + \sin^2 u) + e^{4u}(\sin^2 u + \cos^2 u)} \, du$$

$$\underbrace{-4e^{4u}\sin u\cos u + 4e^{4u}\sin u \cos u}_{0}$$

$$= \sqrt{5}\int_0^t e^{4u}\,du = \frac{\sqrt{5}}{4} e^{4u}\Big|_0^t = \frac{\sqrt{5}}{4}(e^{4t} - 1)$$

$$\therefore \quad s = \frac{\sqrt{5}}{4}(e^{4t} - 1) \longrightarrow \frac{4}{\sqrt{5}} s + 1 = e^{4t}$$

$$t = \frac{1}{4} \ln\left(\frac{4s}{\sqrt{5}} + 1\right)$$

$$\vec{r}(t) = \vec{r}\left(\frac{1}{4}\ln\left(\frac{4s}{\sqrt{5}} + 1\right)\right) = \left\langle \left(\frac{4s}{\sqrt{5}} + 1\right)^{1/2} \cos\left(\frac{1}{4}\ln\left(\frac{4s}{\sqrt{5}} + 1\right)\right), 2, \ldots \right\rangle$$

$$\vec{r}(t) = \langle e^t \cos t \,,\, 2 \,,\, e^t \sin t \rangle$$

$$s(t) = \int_0^t \sqrt{(e^u \cos u - e^u \sin u)^2 + 0^2 + (e^u \sin u + e^u \cos u)^2}\; du$$

$$= \int_0^t \sqrt{e^{2u}\cos^2 u \cancel{- 2e^{2u} \sin u \cos u} + e^{2u} \sin^2 u + e^{2u}\sin^2 u \cancel{+ 2e^{2u} \sin u \cos u} + e^{2u} \cos^2 u}\; du$$

$$= \int_0^t \sqrt{e^{2u} + e^{2u}}\, du = \sqrt{2} \int_0^t e^u \, du = \sqrt{2}\, e^u \Big|_0^t$$

$$s = \sqrt{2}(e^t - 1)$$

$$t = \ln\left(\frac{s}{\sqrt{2}} + 1\right)$$

$$\longrightarrow \vec{r}(t) = \vec{r}\left(\ln\left(\frac{s}{\sqrt{2}} + 1\right)\right)$$

$$= \Big\langle \left(\frac{s}{\sqrt{2}} + 1\right) \cos\left(\ln\left(\frac{s}{\sqrt{2}} + 1\right)\right),\, 2,\, \left(\frac{s}{\sqrt{2}} + 1\right) \sin(\ln(\frac{s}{\sqrt{2}} + 1$$