§10.8 ARC LENGTH

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1. $\vec{r}(t) = \langle t, 3\cos t, 3\sin t \rangle$, $-5 \le t \le 5$

$\vec{r}\,'(t) = \langle 1, -3\sin t, 3\cos t \rangle$

$$L = \int_C ds = \int_{-5}^{5} |\vec{r}\,'(t)|\,dt = \int_{-5}^{5} \sqrt{1 + 9\sin^2 t + 9\cos^2 t}\;dt$$

$$= \sqrt{10}\int_{-5}^{5} dt = \boxed{10\sqrt{10}}$$

3. $\vec{r}(t) = \langle 1, t^2, t^3 \rangle$, $0 \le t \le 1$

$\vec{r}\,'(t) = \langle 0, 2t, 3t^2 \rangle$

$$L = \int_C ds = \int_0^1 |\vec{r}\,'(t)|\,dt = \int_0^1 \sqrt{4t^2 + 9t^4}\;dt$$

$$= \int_0^1 t\sqrt{4 + 9t^2}\;dt$$

LET $u = 4 + 9t^2$

$du = 18t$

$$\rightsquigarrow \frac{1}{18}\int_4^{13} \sqrt{u}\,du = \frac{1}{18}\cdot\frac{2}{3}\,u^{3/2}\Big|_4^{13} = \boxed{\frac{13^{3/2} - 8}{27}}$$

7. C IS INTERSECTION OF $x^2 = 2y$ & $3z = xy$.

PARAMETRIZATION: LET $x = t$. THEN

$$y = \frac{1}{2}t^2 \quad \text{AND}$$

$$z = \frac{1}{6}t^3.$$

SO $\vec{r}(t) = \langle t, \frac{1}{2}t^2, \frac{1}{6}t^3 \rangle$, $0 \le t \le 6$

$\vec{r}(0) = \langle 0,0,0 \rangle$ $\quad$ $\vec{r}(6) = \langle 6, 18, 36 \rangle$

$$\vec{r}\,'(t) = \langle 1, t, \tfrac{1}{2}t^2 \rangle$$

$$L = \int_C ds = \int_0^6 |\vec{r}\,'(t)|\,dt = \int_0^6 \sqrt{1 + t^2 + \tfrac{1}{4}t^4}\,dt$$

$$= \int_0^6 \sqrt{(1 + \tfrac{1}{2}t^2)^2}\,dt = \int_0^6 1 + \tfrac{1}{2}t^2\,dt = t + \tfrac{1}{6}t^3 \Big|_0^6.$$

$$= 6 + 6^2 = \boxed{42}$$

9. REPARAMETRIZE W.R.T. ARCLENGTH s FROM $\vec{r}(0)$.

$$\vec{r}(t) = \langle 2t, 1 - 3t, 5 + 4t \rangle$$

$$\vec{r}\,'(t) = \langle 2, -3, 4 \rangle$$

$$s = \int_0^t |\vec{r}\,'(u)|\,du = \int_0^t \sqrt{4 + 9 + 16}\,du = \sqrt{29}\,t$$

$$\Rightarrow t = \frac{1}{\sqrt{29}}s \quad \Rightarrow \vec{r}(t) = \vec{r}\left(\frac{1}{\sqrt{29}}s\right) = \boxed{\left\langle \frac{2}{\sqrt{29}}s, 1 - \frac{3}{\sqrt{29}}s, 5 + \frac{4}{\sqrt{29}}s \right\rangle}$$

11. $\vec{r}(t) = \langle 3\sin t, 4t, 3\cos t \rangle$

$\vec{r}(0) = \langle 0, 0, 3 \rangle$

$$s = \int_0^t |\vec{r}\,'(u)|\,du = \int_0^t \sqrt{9\cos^2 t + 16 + 9\sin^2 t}\;dt$$

$\therefore\ s = 5t \quad \Rightarrow \quad t = \frac{1}{5}s$

SO WHEN $s = 5$, $t = 1$

$$\vec{r}(1) = \boxed{\langle 3\sin(1), 4, 3\cos(1) \rangle}$$

12. $\vec{r}(t) = \left\langle \frac{2}{t^2+1} - 1, \frac{2t}{t^2+1} \right\rangle$, NOTE: $\vec{r}(0) = \langle 1, 0 \rangle$

$$\vec{r}\,'(t) = \left\langle \frac{-4t}{(t^2+1)^2}, \frac{2(t^2+1) - 4t^2}{(t^2+1)^2} \right\rangle$$

$$= \left\langle \frac{-4t}{(t^2+1)^2}, \frac{2-2t^2}{(t^2+1)^2} \right\rangle$$

$$s = \int_0^t |\vec{r}\,'(u)|\,du = \int_0^t \sqrt{\frac{16u^2 + 4 - 8u^2 + 4u^4}{(u^2+1)^4}}\;du$$

$$= \int_0^t \sqrt{\frac{4 + 8u^2 + 4u^4}{(u^2+1)^4}}\;d = \int_0^t \frac{2+2u^2}{(u^2+1)^2}\;du$$

$$= 2\int_0^t \frac{1}{u^2+1}\;du = 2\tan^{-1} u \Big|_0^t = 2\tan^{-1} t$$

$\therefore \quad s = 2\tan^{-1} t$

$t = \tan\left(\frac{1}{2}s\right)$

$$\Rightarrow \vec{r}(t) = \vec{r}\left(\tan\left(\tfrac{1}{2}s\right)\right)$$

$$= \left\langle \frac{2}{\tan^2\left(\frac{1}{2}s\right)+1} - 1 , \frac{2\tan\left(\frac{1}{2}s\right)}{\tan^2\left(\frac{1}{2}s\right)+1} \right\rangle$$

$$= \left\langle \frac{2}{\sec^2\left(\frac{1}{2}s\right)} - 1 , \frac{2\tan\left(\frac{1}{2}s\right)}{\sec^2\left(\frac{1}{2}s\right)} \right\rangle$$

$$= \left\langle 2\cos^2\left(\tfrac{1}{2}s\right) - 1 , 2\sin\left(\tfrac{1}{2}s\right)\cos\left(\tfrac{1}{2}s\right) \right\rangle$$

$$= \boxed{\langle \cos(s) , \sin(s) \rangle}$$

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