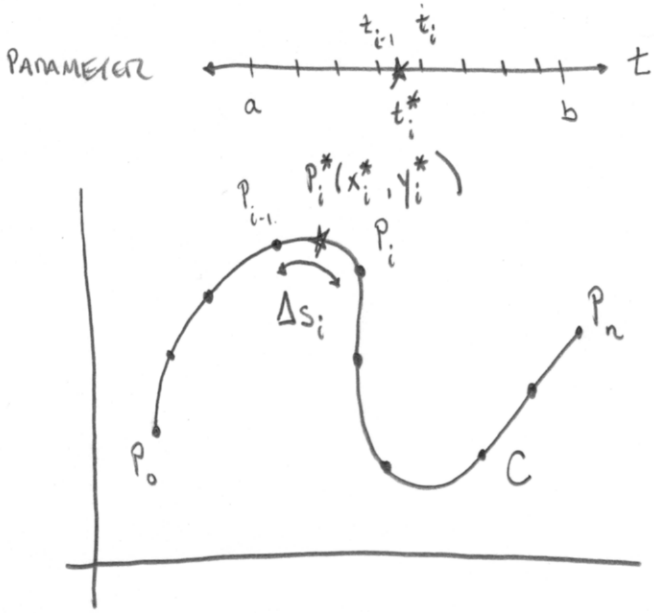
§13.2 LINE INTEGRALS

INSTEAD OF INTEGRATING A FUNCTION OF 1 VAR. OVER AN INTERVAL,
WE INTEGRATE A FUNCTION OF SEVERAL VAR. OVER A CURVE, C.



C DESCRIBED BY

$\vec{r}(t) = \langle x(t), y(t) \rangle$

i) $\vec{r}\,'$ CONTINUOUS
ii) $\vec{r}\,' \neq 0$

} SMOOTH CURVE

$P_i(x_i, y_i) = (x(t_i), y(t_i))$

$P_i^*(x_i^*, y_i^*) = (x(t_i^*), y(t_i^*))$

SAMPLE POINT IN t^{TH} SUBINTERVAL OF $[a, b]$

SAMPLE POINT IN t^{TH} SUBARC OF C

Δs_i = LENGTH OF i^{TH} SUBARC

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i$$

$$\int_C f(x,y)\,ds = \lim_{\max \Delta s_i \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i$$

RECALL: $s(t) = \int_a^t |\vec{r}\,'(u)|\,du$

$$\Rightarrow \frac{ds}{dt} = |\vec{r}\,'(t)| \quad \Rightarrow \quad ds = |\vec{r}\,'(t)|\,dt$$

$$ds = \sqrt{x'(t)^2 + y'(t)^2}\,dt$$

$$\boxed{\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{x'(t)^2 + y'(t)^2}\,dt}$$

IF $f \geq 0$ ALONG C, THEN "AREA OF CURTAIN" (ONE SIDE)

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ex. $\int_C xy^4\,ds$, C IS RIGHT HALF-CIRCLE $x^2+y^2=16$

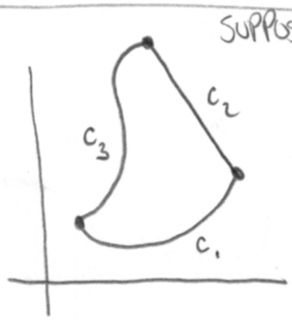
$x = \cos t \qquad y = \sin t, \qquad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$

$x'(t) = -\sin t \qquad y'(t) = \cos t$

$$\to \int_{-\pi/2}^{\pi/2} \cos t \sin^4 t \sqrt{(-\sin t)^2 + (\cos t)^2}\,dt \sim \int_{-1}^{1} u^4\,du$$

$$= \frac{1}{5}u^5 \Big|_{-1}^{1} = \frac{1}{5}(1-(-1)) = \frac{2}{5}$$

SUPPOSE C IS PIECEWISE SMOOTH CURVE.



$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \int_{C_3} f(x,y)\,ds$$

ANOTHER INTERPRETATION : SAY $\rho(x,y)$ GIVES DENSITY OF A WIRE (CURVE) AT POINT (x,y).

THEN $m = \int_C \rho(x,y)\,ds$ IS THE MASS OF THE WIRE.

FURTHERMORE, CENTER OF MASS :

$$\bar{x} = \frac{1}{m}\int_C x\,\rho(x,y)\,ds \quad , \quad \bar{y} = \frac{1}{m}\int_C y\,\rho(x,y)\,ds$$

ex. A WIRE TAKES THE SHAPE OF A SEMICIRCLE $x^2+y^2=1$, $y \geq 0$. FIND CENTER OF MASS IF LINEAR DENSITY IS PROPORTIONAL TO DISTANCE FROM $y=1$.

$$\rho(x,y) = k(1-y)$$

$x = \cos t \qquad x' = -\sin t$

$y = \sin t \qquad y' = \cos t$

$0 \leq t \leq \pi$

$$m = \int_C k(1-y)\,ds$$

$$= \int_0^{\pi} k(1-\sin t)\sqrt{1}\,dt$$

$$= k(t+\cos t)\Big|_0^{\pi} = k(\pi-2)$$

$\bar{x} = 0$, $\bar{y} = \frac{1}{\cancel{k}(\pi-2)}\int_0^{\pi} \cancel{k}\sin t\,(1-\sin t)\,dt = \frac{1}{\pi-2}\int_0^{\pi} \sin t - \frac{1}{2}(1-\cos 2t)\,dt$

$$= \frac{1}{\pi-2}\left[-\cos t - \frac{1}{2}t + \frac{1}{4}\sin 2t\right]_0^{\pi} = \frac{1}{\pi-2}\left(2-\frac{\pi}{2}\right) = \frac{4-\pi}{2(\pi-2)}$$

LINE INTEGRALS WITH RESPECT TO x & y: RIEMANN SUMS WITH Δx_i, Δy_i LEAD TO.

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$$

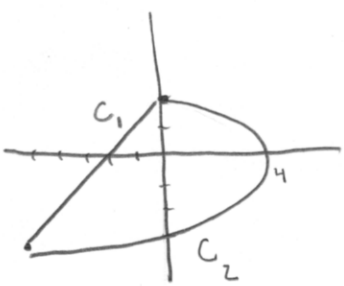
THESE OFTEN COME IN PAIRS: WE WRITE

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_C P(x,y)\,dx + Q(x,y)\,dy$$

ex. $\int_C y^2\,dx + x\,dy$

(a) $C = C_1$ LINE SEGMENT FROM $(-5,-3)$ TO $(0,2)$

(b) $C = C_2$ ARC OF PARABOLA $x = 4 - y^2$ FROM $(-5,-3)$ TO $(0,2)$



(a) $-\frac{5}{6}$ (b) $40\frac{5}{6}$

NOTE: IN GENERAL, LINE INTEGRALS DEPEND ON PATH.

LINE INTEGRALS IN SPACE

ex. $\int_C y \sin z\,ds$, C: $x = \cos t$, $y = \sin t$, $z = t$, $0 \le t \le 2\pi$ CIRCULAR HELIX $(\sqrt{2}\pi)$

ex. $\int_C y\,dx + z\,dy + x\,dz$, $C = C_1 \oplus C_2$

C_1: $(2,0,0)$ TO $(3,4,5)$ 24.5

C_2: $(3,4,5)$ TO $(3,4,0)$ -15

§13.2 LINE INTEGRALS (CONTINUED)

LINE INTEGRALS OF VECTOR FIELDS (IN $\mathbb{R}^3$, $\mathbb{R}^2$ IS SIMILAR)

FOR NOW, SUPPOSE $\vec{F}(\vec{x}) = \vec{F}(x,y,z) = \langle 1, 0, 2 \rangle$ CONSTANT FORCE FIELD.

AND SUPPOSE A PARTICLE MOVES ALONG A LINE SEGMENT

$\vec{r}(t) = \langle 2+4t, 3-5t, 1+3t \rangle$, $t_0 \le t \le t_1$

THE WORK DONE BY THE FORCE FIELD ON THE PARTICLE IS

$$\vec{F} \cdot \langle \text{DISPLACEMENT VECTOR} \rangle$$

$$F \cdot \langle \text{VELOCITY VECTOR} \rangle (\text{TIME})$$

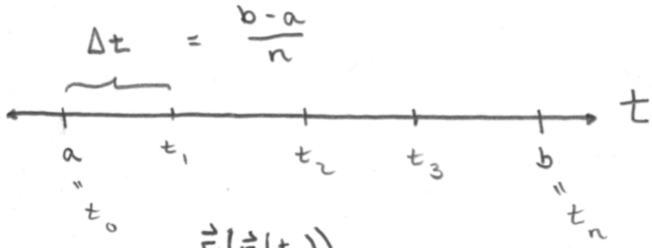
$$\langle 1, 0, 2 \rangle \cdot \langle 4, -5, 3 \rangle (t_1 - t_0)$$

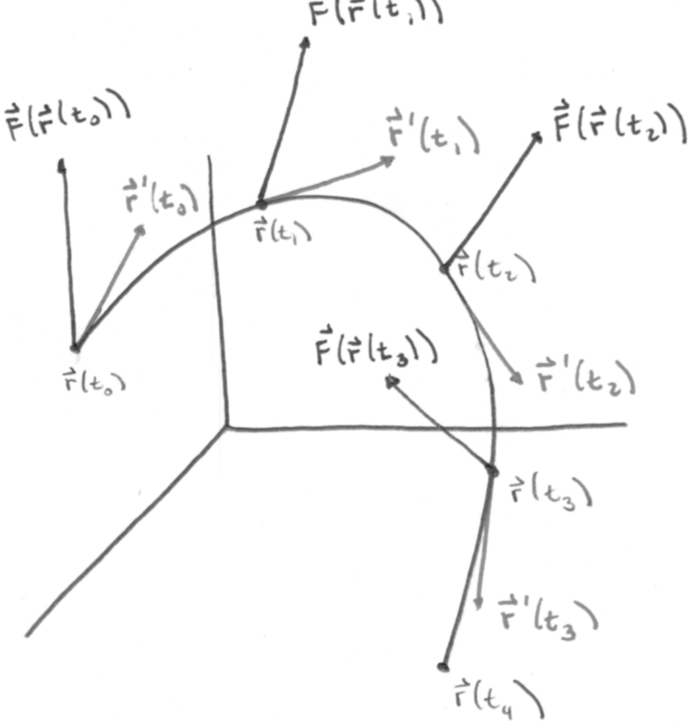
$$\vec{F}(\vec{r}(t_0)) \cdot \vec{r}\,'(t_0)\, \Delta t$$

NOW LET $\vec{F}(\vec{x}) = \vec{F}(x,y,z)$ BE ANY CONTINUOUS VECTOR FIELD,

AND PARTICLE MOVES ALONG A SMOOTH CURVE C WITH POSITION

GIVEN BY $\vec{r}(t)$, $a \le t \le b$





SINCE $\vec{r}$ IS SMOOTH, $\vec{r}\,'$ DOES NOT VARY MUCH FROM t_0 TO t_1. SO WE TREAT THE PARTICLE AS IF IT WERE TRAVELING IN A STRAIGHT LINE AND SAY THAT ITS DISPLACEMENT IS APPROXIMATELY

$$\vec{r}\,'(t_0)\,\Delta t$$

SINCE $\vec{F}$ IS CONTINUOUS, $\vec{F}$ DOES NOT VARY MUCH OVER SHORT DISTANCES. IN PARTICULAR WE CAN APPROXIMATE $\vec{F}$ ALONG THE SUB-ARC OF C FROM $\vec{r}(t_0)$ TO $\vec{r}(t_1)$ BY $\vec{F}(\vec{r}(t_0))$.

THUS, THE WORK DONE BY THE VECTOR FIELD ON THE PARTICLE AS THE PARTICLE MOVES FROM $\vec{r}(t_i)$ TO $\vec{r}(t_{i+1})$ IS APPROXIMATELY

$$W_i \approx \vec{F}(\vec{r}(t_i)) \cdot \vec{r}\,'(t_i)\,\Delta t$$

AND THE TOTAL WORK ALONG C IS $W \approx \sum_{i=0}^{n-1} \vec{F}(\vec{r}(t_i)) \cdot \vec{r}\,'(t_i)\,\Delta t$

WE DEFINE

$$W = \int_C \vec{F} \cdot d\vec{r} = \lim_{n\to\infty} \sum_{i=0}^{n-1} \vec{F}(\vec{r}(t_i)) \cdot \vec{r}\,'(t_i)\,\Delta t$$

THAT IS, $\boxed{\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt}$

NOTE: $\dfrac{d\vec{r}}{dt} = \vec{r}\,'(t) \Rightarrow d\vec{r} = \vec{r}\,'(t)\,dt$

ex. EVALUATE $\int_C \vec{F}\cdot d\vec{r}$ WHERE $\vec{F}(x,y,z) = \langle x+y,\ y-z,\ z^2\rangle$

AND $\vec{r}(t) = \langle t^2, t^3, t^2\rangle$, $0 \le t \le 1$.

ex. FORCE EXERTED BY ELECTRIC CHARGE AT ORIGIN ON CHARGED PARTICLE AT THE POINT (x,y,z) WITH POSITION VECTOR $\vec{r} = \langle x,y,z\rangle$ IS $\vec{F}(\vec{r}) = \dfrac{K\vec{r}}{|\vec{r}\,|^3}$. K CONSTANT.

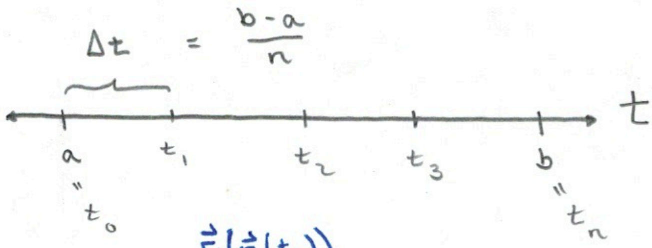
FIND WORK DONE AS PARTICLE MOVES ALONG STRAIGHT LINE FROM $(2,0,0)$ TO $(2,1,5)$.

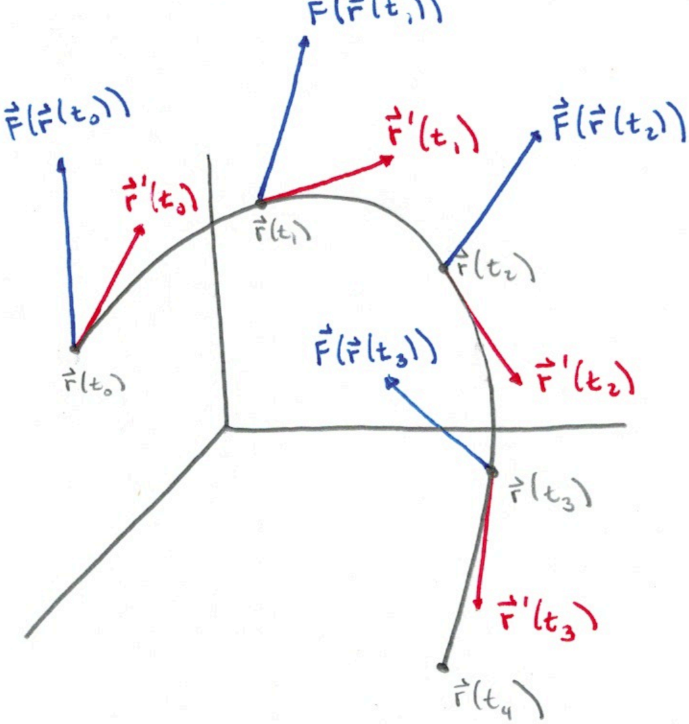
CONNECTION BETWEEN LINE INTEGRALS OF VECTOR FIELDS VS. SCALAR FIELDS

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle \cdot \langle x'(t), y'(t), z'(t)\rangle\,dt$$

$$= \int_a^b P(x,y,z)\,x'(t)\,dt + Q(x,y,z)\,y'(t)\,dt + R(x,y,z)\,z'(t)\,dt$$

$$= \int_a^b P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$$





Since $\vec{r}$ is smooth, $\vec{r}\,'$ does not vary much from t_0 to t_1. So we treat the particle as if it were traveling in a straight line and say that its displacement is approximately

$$\vec{r}\,'(t_0)\,\Delta t$$

Since $\vec{F}$ is continuous, $\vec{F}$ does not vary much over short distances. In particular we can approximate $\vec{F}$ along the sub-arc of C from $\vec{r}(t_0)$ to $\vec{r}(t_1)$ by $\vec{F}(\vec{r}(t_0))$.

Thus, the work done by the vector field on the particle as the particle moves from $\vec{r}(t_i)$ to $\vec{r}(t_{i+1})$ is approximately

$$W_i \approx \vec{F}(\vec{r}(t_i)) \cdot \vec{r}\,'(t_i)\,\Delta t$$

and the total work along C is $W \approx \displaystyle\sum_{i=0}^{n-1} \vec{F}(\vec{r}(t_i)) \cdot \vec{r}\,'(t_i)\,\Delta t$

We define

$$W = \int_C \vec{F} \cdot d\vec{r} = \lim_{n\to\infty} \sum_{i=0}^{n-1} \vec{F}(\vec{r}(t_i)) \cdot \vec{r}\,'(t_i)\,\Delta t$$