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4. $\int_C x \sin y \, ds$, C LINE SEGMENT FROM $(0,3)$ TO $(4,6)$

PARAMETRIZE THE CURVE : $\vec{r}(t) = (1-t)\langle 0,3\rangle + t\langle 4,6\rangle$, $0 \le t \le 1$

$$\vec{r}(t) = \langle 4t, 3+3t\rangle$$

$$\vec{r}\,'(t) = \langle 4, 3\rangle$$

$x = 4t$, $y = 3+3t$

$$ds = |\vec{r}\,'(t)|\,dt = \sqrt{4^2+3^2}\,dt = 5\,dt$$

$$\int_C x \sin(y)\,ds = \int_0^1 (4t)\sin(3+3t)\cdot 5\,dt$$

$$= 20\int_0^1 t\sin(3+3t)\,dt$$

INT. BY PARTS :

$u = t$ $\quad$ $v = -\frac{1}{3}\cos(3+3t)$

$du = dt$ $\quad$ $dv = \sin(3+3t)\,dt$

$$= 20\left[-\frac{1}{3}t\cos(3+3t)\Big|_0^1 + \frac{1}{3}\int_0^1 \cos(3+3t)\,dt\right]$$

$$= 20\left[-\frac{1}{3}\cos(6) + \frac{1}{9}\sin(3+3t)\Big|_0^1\right]$$

$$= 20\left[-\frac{1}{3}\cos(6) + \frac{1}{9}\left(\sin(6) - \sin(3)\right)\right]$$

$$= \boxed{\frac{20}{9}\left(-3\cos(6) + \sin(6) - \sin(3)\right)}$$

5. $\int_C (x^2y^3 - \sqrt{x})\, dy$

C IS CURVE $y = \sqrt{x}$ FROM $(1,1)$ TO $(4,2)$

PARAMETRIZE : $x = t \quad , \quad y = \sqrt{t} \quad , \quad 1 \le t \le 4$

$$dy = \frac{1}{2\sqrt{t}}\, dt$$

$$\int_C (x^2y^3 - \sqrt{x})\, dy = \int_1^4 \left((t^2)(\sqrt{t})^3 - \sqrt{t} \right) \frac{1}{2\sqrt{t}}\, dt$$

$$= \frac{1}{2} \int_1^4 t^3 - 1\, dt = \frac{1}{2}\left[\frac{1}{4}t^4 - t \right]_1^4$$

$$= \frac{1}{2}\left[\frac{1}{4}(4^4 - 1) - (4-1) \right] = \frac{1}{2}\left(\frac{255}{4} - 3 \right) = \boxed{\frac{243}{8}}$$

7. $\int_C (x+2y)\, dx + x^2\, dy \quad , \quad C = C_1 \oplus C_2$

$C_1 : \quad x = 2t \quad y = t \quad , \quad 0 \le t \le 1$

$dx = 2\, dt \quad dy = dt$

$$\Rightarrow \int_{C_1} (x+2y)\, dx + x^2\, dy = \int_0^1 \left((2t + 2t)\cdot 2 + (2t)^2 \right) dt$$

$$= \int_0^1 (8t + 4t^2)\, dt = 4t^2 + \frac{4}{3}t^3 \Big|_0^1 = \underline{\underline{\frac{16}{3}}}$$

$C_2: \quad x = 2 + t \qquad y = 1 - t \quad , \quad 0 \le t \le 1$

$dx = dt \qquad dy = -dt$

$$\int_{C_2} (x+2y)\,dx + x^2\,dy = \int_0^1 \Big((2+t) + 2(1-t) - (2+t)^2 \Big)\,dt$$

$$= \int_0^1 \left(-5t - t^2\right) dt = -\frac{5}{2}t^2 - \frac{1}{3}t^3 \Big|_0^1 = \frac{-15-2}{6} = -\frac{17}{6}$$

$$\int_C = \int_{C_1} + \int_{C_2} = \frac{16}{3} - \frac{17}{6} = \frac{32-17}{6} = \frac{15}{6} = \boxed{\frac{5}{2}}$$

9. $\int_C xyz\,ds \ , \quad C: \ \vec{r}(t) = \langle 2\sin t \ , t \ , -2\cos t \rangle \ , \ 0 \le t \le \pi$

$$\vec{r}\,'(t) = \langle 2\cos t \ , 1 \ , 2\sin t \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{4\cos^2 t + 1 + 4\sin^2 t} = \sqrt{5}$$

$$= \int_0^{\pi} x(t)\,y(t)\,z(t)\,|\vec{r}\,'(t)|\,dt$$

$$= \int_0^{\pi} -4t\,\underbrace{\sin(t)\cos(t)}_{\frac{1}{2}\sin(2t)}\sqrt{5}\,dt = -2\sqrt{5}\int_0^{\pi} t\sin(2t)\,dt$$

$u = t \qquad v = -\frac{1}{2}\cos(2t)$

$du = dt \qquad dv = \sin(2t)\,dt$

$$\longrightarrow -2\sqrt{5}\left[-\frac{1}{2}t\cos(2t)\Big|_0^{\pi} + \frac{1}{2}\int_0^{\pi}\cos(2t)\,dt\right.$$

$$= -2\sqrt{5}\left[-\frac{\pi}{2} + \underbrace{\frac{1}{4}\sin(2t)\Big|_0^{\pi}}_{0}\right] = \boxed{\sqrt{5}\,\pi}$$

11. $\int_C xe^{yz}\,ds \qquad C: \vec{r}(t) = \langle t, 2t, 3t\rangle, \quad 0 \le t \le 1$

$$\vec{r}\,'(t) = \langle 1, 2, 3\rangle$$

$$|\vec{r}\,'(t)| = \sqrt{1+4+9} = \sqrt{14}$$

$$= \int_0^1 te^{6t^2}\sqrt{14}\,dt \qquad \text{LET } u = 6t^2$$

$$du = 12t\,dt$$

$$\rightsquigarrow \frac{\sqrt{14}}{12}\int_0^6 e^u\,du = \frac{\sqrt{14}}{12}e^u\Big|_0^6 = \boxed{\frac{\sqrt{14}}{12}\left(e^6 - 1\right)}$$

15. $\int_C z^2\,dx + x^2\,dy + y^2\,dz$

$$x = 1+3t, \quad y = t, \quad z = 2t, \quad 0 \le t \le 1$$

$$dx = 3\,dt \qquad dy = dt \qquad dz = 2\,dt$$

$$\int_C = \int_0^1 \left((2t)^2 \cdot 3 + (1+3t)^2 + t^2 \cdot 2 \right) dt$$

$$12t^2 + 1 + 6t + 9t^2 + 2t^2$$

$$= \int_0^1 \left(23t^2 + 6t + 1\right) dt = \frac{23}{3} t^3 + 3t^2 + t \Big|_0^1$$

$$= \frac{23 + 9 + 3}{3} = \boxed{\frac{35}{3}}$$

17. (a) $\boxed{\text{POSITIVE}}$

NOTE THAT $\vec{F} \cdot d\vec{r} = \underbrace{\vec{F} \cdot \vec{r}\,'}\, dt$

$$-\frac{\pi}{2} < \text{ANGLE BETWEEN THESE VECTORS } \underset{\theta}{} < \frac{\pi}{2}$$

$$\text{so } \vec{F} \cdot \vec{r}\,' = |\vec{F}| \cdot |\vec{r}\,'| \cos\theta > 0$$

(b) $\boxed{\text{NEGATIVE}}$

NOW $\vec{F}$ & $\vec{r}\,'$ POINT IN OPPOSITE DIRECTIONS !

$$\vec{F} \cdot \vec{r}\,' = |\vec{F}| \cdot |\vec{r}\,'| \underbrace{\cos(\pi)}_{-1}$$

19. $\vec{F}(x,y) = \langle xy\ ,\ 3y^2 \rangle$

$\vec{r}(t) = \langle 11t^4\ ,\ t^3 \rangle \quad , \quad 0 \le t \le 1$

$\vec{r}\,'(t) = \langle 44t^3\ ,\ 3t^2 \rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(t) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^1 \langle 11t^7\ ,\ 3t^6 \rangle \cdot \langle 44t^3\ ,\ 3t^2 \rangle\, dt$$

$$= \int_0^1 484t^{10} + 9t^8\, dt = 44t^{11} + t^9 \Big|_0^1 = \boxed{45}$$

21. $\vec{F}(x,y,z) = \langle \sin x\ ,\ \cos y\ ,\ xz \rangle$

$\vec{r}(t) = \langle t^3\ ,\ -t^2\ ,\ t \rangle$

$\vec{r}\,'(t) = \langle 3t^2\ ,\ -2t\ ,\ 1 \rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(t) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^1 \langle \sin(t^3)\ ,\ \cos(-t^2)\ ,\ t^4 \rangle \cdot \langle 3t^2, -2t\ ,\ 1 \rangle\, dt$$

$$= \int_0^1 \left(3t^2 \sin(t^3) - 2t \cos(-t^2) + t^4 \right) dt$$

$$= -\cos(t^3) + \sin(-t^2) + \frac{1}{5}t^5 \Big|_0^1$$

$$= -\cos(1) + 1 + \sin(-1) + \frac{1}{5} = \boxed{\frac{6}{5} - \cos(1) - \sin(1)}$$

31. BY SYMMETRY, $\bar{y} = 0$.

$$\bar{x} = \frac{1}{m} \int_C x \rho(x,y)\, ds \quad \text{WHERE} \quad m = \int_C \rho(x,y)\, ds$$

$$\rho(x,y) = k, \quad C: \ x = 2\cos t, \quad y = 2\sin t, \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

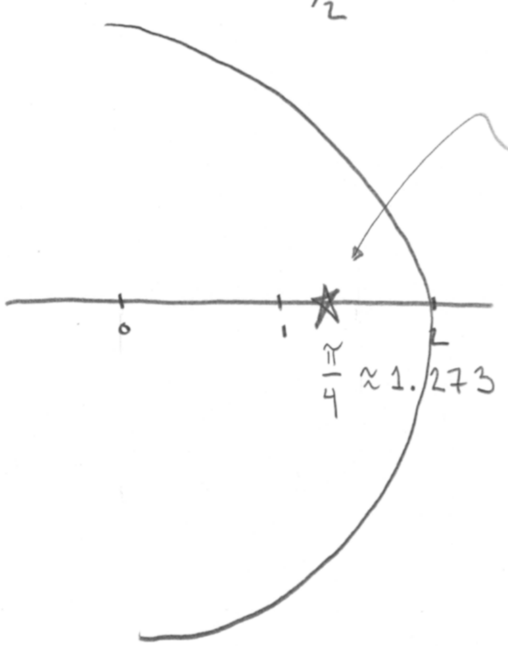
$$x'(t) = -2\sin t \qquad y'(t) = 2\cos t$$

$$|\vec{r}\,'(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2} = 2$$

$$m = \int_{-\pi/2}^{\pi/2} 2k\, dt = 2kt \Big|_{-\pi/2}^{\pi/2} = 2\pi k \quad (\text{LENGTH} \times \text{DENSITY})$$

$$\bar{x} = \frac{1}{2\pi k} \int_{-\pi/2}^{\pi/2} 2\cos t \cdot k \cdot 2\, dt = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} \cos t\, dt$$

$$= \frac{2}{\pi} \sin t \Big|_{-\pi/2}^{\pi/2} = \frac{4}{\pi} \qquad \therefore \ \boxed{(\bar{x}, \bar{y}) = \left(\frac{4}{\pi}, 0\right) \quad m = 2\pi k}$$



37. $\vec{F}(x,y) = \langle x, y+2 \rangle$

$\vec{r}(t) = \langle t - \sin t, 1 - \cos t \rangle, \quad 0 \le t \le 2\pi$

$\vec{r}\,'(t) = \langle 1 - \cos t, \sin t \rangle$

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^{2\pi} \langle t - \sin t, 3 - \cos t \rangle \cdot \langle 1 - \cos t, \sin t \rangle\, dt$$

$$t - t\cos t - \sin t + \cancel{\sin t \cos t} + 3\sin t - \cancel{\sin t \cos t}$$

$$= \int_0^{2\pi} (t + 2\sin t - t\cos t)\, dt$$

$$= \left[\frac{1}{2}t^2 - 2\cos t\right]_0^{2\pi} - \int_0^{2\pi} t\cos t\, dt \qquad \begin{array}{ll} u = t & v = \sin t \\ du = dt & dv = \cos t\, dt \end{array}$$

$$= 2\pi^2 - \left[\underbrace{t\sin t\,\Big|_0^{2\pi}}_{0} - \underbrace{\int_0^{2\pi} \sin t\, dt}_{0}\right] = \boxed{2\pi^2}$$

39. $\vec{F}(x,y,z) = \langle x - y^2, y - z^2, z - x^2 \rangle$

$\vec{r}(t) = \langle 2t, t, 1 - t \rangle, \quad 0 \le t \le 1$

$\vec{r}\,'(t) = \langle 2, 1, -1 \rangle$

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(x(t), y(t), z(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^1 \langle 2t - t^2, \underbrace{t - (1-t)^2}_{-1+3t-t^2}, 1 - t - 4t^2 \rangle \cdot \langle 2, 1, -1 \rangle \, dt$$

$$= \int_0^1 2(2t - t^2) + (-1 + 3t - t^2) - (1 - t - 4t^2)\, dt$$

$$4t - 2t^2 - 1 + 3t - t^2 - 1 + t + 4t^2$$

$$= \int_0^1 \left(t^2 + 8t - 2\right) dt = \frac{1}{3}t^3 + 4t^2 - 2t \Big|_0^1 = \boxed{\frac{7}{3}}$$

43. $\vec{F}(x, y, z) = \langle 0, 0, -185 \rangle$

$\vec{r}(t) = \langle 20\cos(6\pi t), 20\sin(6\pi t), 90t \rangle, \quad 0 \le t \le 1$

$\vec{r}\,'(t) = \langle -120\pi \sin(6\pi t), 120\pi\cos(6\pi t), 90 \rangle$

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_0^1 \langle 0, 0, -185 \rangle \cdot \langle -120\pi\sin(6\pi t), 120\pi\cos(6\pi t), 90 \rangle\, dt$$

$$= \int_0^1 -185 \cdot 90\, dt = -16650\, t \Big|_0^1 = -16{,}650 \text{ ft} \cdot \text{lbs.}$$

GRAVITY DOES NEG. WORK
THE MAN DOES POS. WORK

(NOTE: YOU DON'T NEED CALCULUS TO ANSWER THIS QUESTION!)

44. WEIGHT IS 185 LBS AT BOTTOM $(t=0)$

WEIGHT IS 176 LBS AT TOP $(t=1)$

SO $\vec{F}(\vec{r}(t)) = \langle 0, 0, 185-9t \rangle$

$$W = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_0^1 (185-9t) \cdot 90\,dt = 90\left[185t - \frac{9}{2}t^2\right]_0^1$$

$$= 16{,}650 - 405 = \boxed{16245 \text{ ft} \cdot \text{LBS}}$$

46. $C: \vec{r}(t), \quad a \le t \le b$

$$\int_C \vec{r} \cdot d\vec{r} = \int_a^b \vec{r}(t) \cdot \vec{r}\,'(t)\,dt$$, ASSUMING DIMENSION = 3 :

$$= \int_a^b \Big(x(t)x'(t) + y(t)y'(t) + z(t)z'(t)\Big)\,dt$$

$$= \left(\frac{1}{2}x(t)^2 + \frac{1}{2}y(t)^2 + \frac{1}{2}z(t)^2\right)\Bigg|_a^b$$

$$= \frac{1}{2}\Big(\big(x(b)^2 + y(b)^2 + z(b^2)\big) - \big(x(a)^2 + y(a)^2 + z(a)^2\big)\Big)$$

$$= \frac{1}{2}\left[\,|\vec{r}(b)|^2 - |\vec{r}(a)|^2\,\right]$$

✓

47. (a) LET $\vec{F}(x,y,z) = \langle a, b \rangle$ (CONSTANTS)

$\vec{r}(t) = \langle \cos t , \sin t \rangle , \quad 0 \le t \le 2\pi$

$\vec{r}\,'(t) = \langle -\sin t , \cos t \rangle$

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle a, b \rangle \cdot \langle -\sin t , \cos t \rangle \, dt$$

$$= \int_0^{2\pi} \left(-a \sin t + b \cos t \right) dt = 0 \quad \checkmark$$

(b) LET $\vec{F}(x,y) = k \langle x, y \rangle = \langle kx , ky \rangle$

$$W = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \, dt$$

$$= \int_0^{2\pi} \langle k \cos t , k \sin t \rangle \cdot \langle -\sin t , \cos t \rangle \, dt$$

$$= k \int_0^{2\pi} -\sin t \cos t + \sin t \cos t \; dt = 0 \quad \checkmark$$